MOMENT OF A FORCE (SCALAR FORMULATION), CROSS PRODUCT, MOMENT OF A FORCE (VECTOR FORMULATION), & PRINCIPLE OF MOMENTS

Objectives :

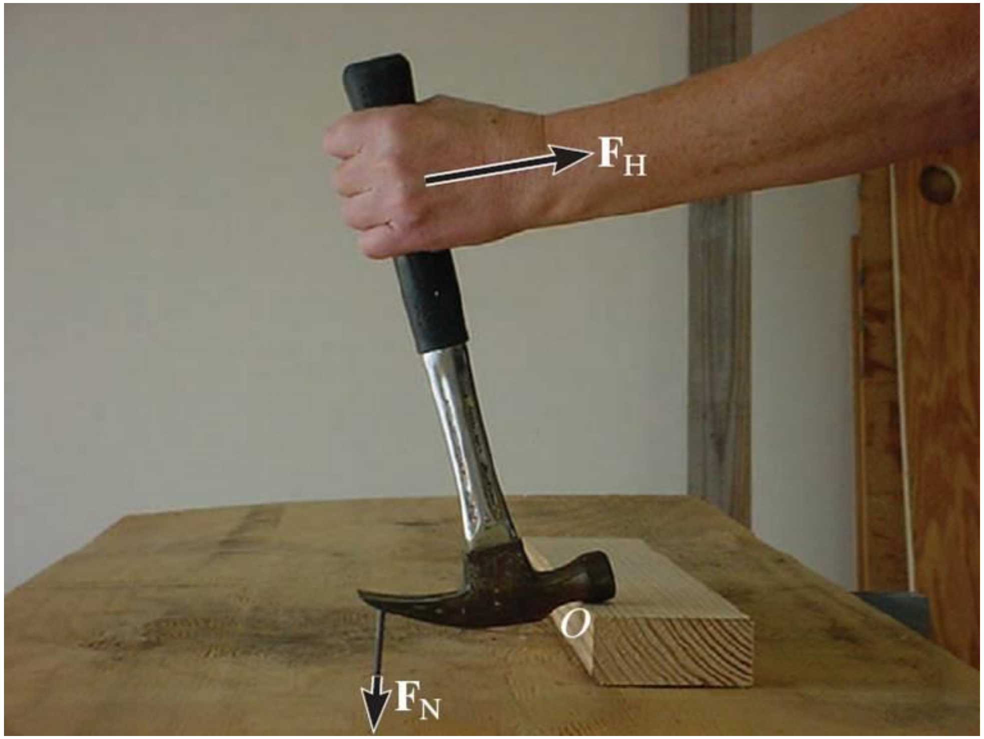
Students will be able to:

a) understand and define moment,

b) determine moments of a force in 2-D and 3-D cases, and

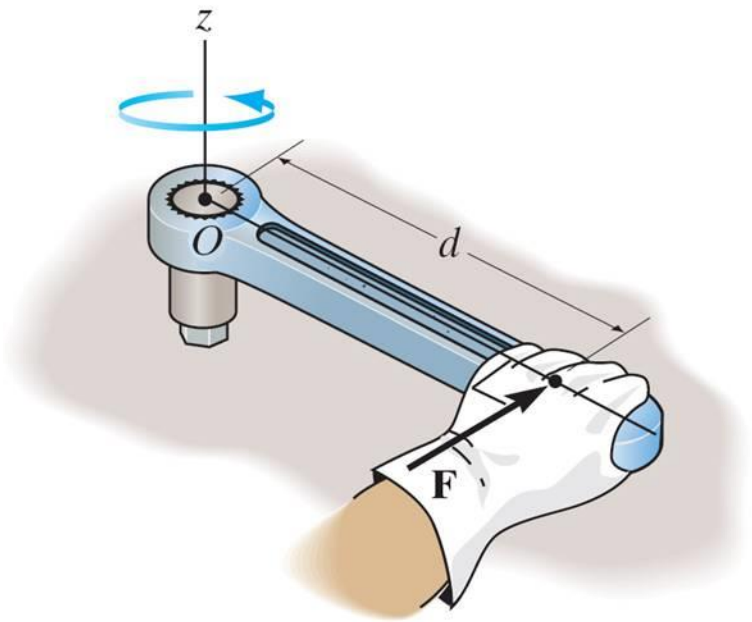
c) determine the moment of a force about an axis

APPLICATIONS (continued)



Carpenters often use a hammer in this way to pull a stubborn nail. Through what sort of action does the force F_H at the handle pull the nail? How can you mathematically model the effect of force F_H at point O?

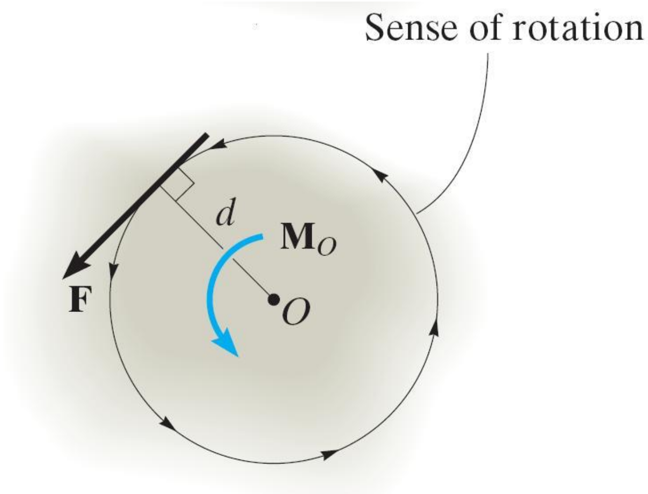
MOMENT OF A FORCE - SCALAR FORMULATION (Section 4.1)



The moment of a force about a point provides a measure of the tendency for rotation (sometimes called a torque).

MOMENT OF A FORCE - SCALAR FORMULATION (continued)

In a 2-D case, the magnitude of the moment is $M_o = F\ d$

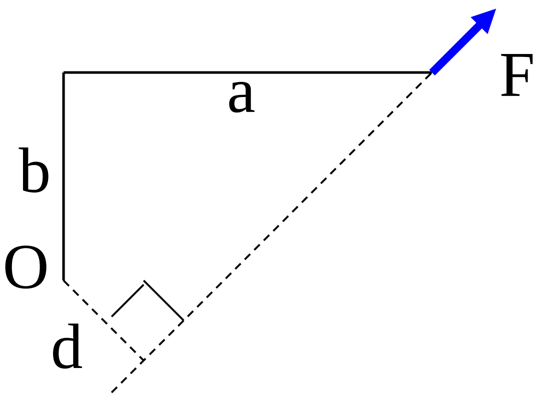


Moment Units are:
N.m or lb.ft

As shown, d is the perpendicular distance from point O to the line of action of the force.

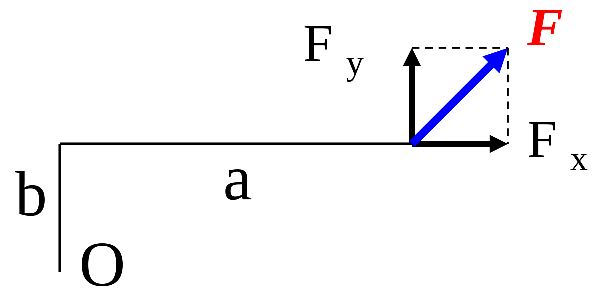
In 2-D, the direction of M_O is either clockwise (CW) or counter-clockwise (CCW), depending on the tendency for rotation.

MOMENT OF A FORCE - SCALAR FORMULATION (continued)



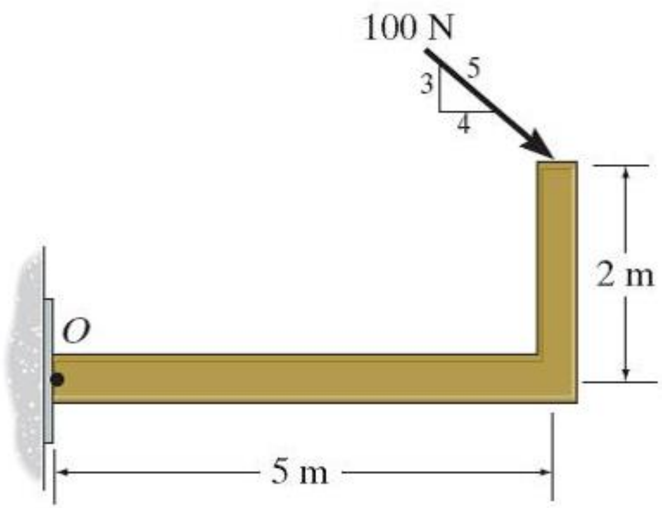
For example, M_O = F d and the direction is counter-clockwise.

Often it is easier to determine M_O by using the components of ***F*** as shown.



Then $M_O = (F_y\ a) - (F_x\ b)$. Note the different signs on the terms! The typical sign convention for a moment in 2-D is that counter-clockwise is considered positive. We can determine the direction of rotation by imagining the body pinned at O and deciding which way the body would rotate because of the force.

EXAMPLE I



Given: A 100 N force is applied to the frame.

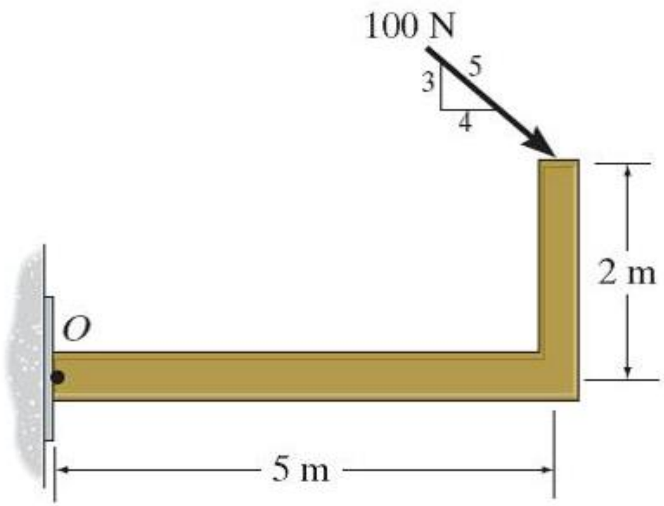
Find: The moment of the force at point O.

Plan:

1) Resolve the 100 N force along x and y-axes.

2) Determine M_O using a scalar analysis for the two force components and then add those two moments together.

EXAMPLE I (continued)



Solution:

$+\uparrow F_y = \underline{-\,100\,(3/5)\text{ N}}$

$+\rightarrow F_x = \underline{100\,(4/5)\text{ N}}$

$+\circlearrowleft M_O = \{-\,100\,(3/5)\text{N}\,(5\text{ m}) - (100)(4/5)\text{N}\,(2\text{ m})\}\ \text{N}\cdot\text{m}$

$= \underline{-\,460\ \text{N}\cdot\text{m}}\circlearrowleft$ or $\underline{460\ \text{N}\cdot\text{m CW}}$

PEARSON

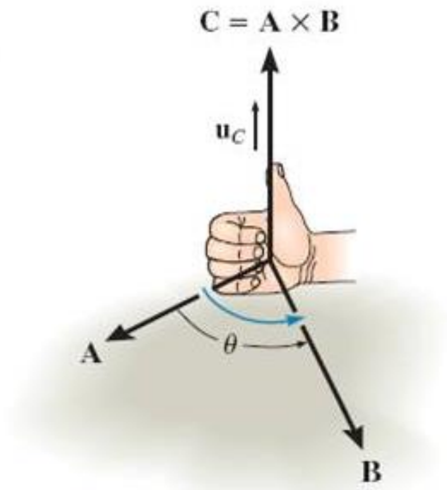
VECTOR CROSS PRODUCT (Section 4.2)

While finding the moment of a force in 2-D is straightforward when you know the perpendicular distance d, finding the perpendicular distances can be hard—especially when you are working with forces in three dimensions.

So a more general approach to finding the moment of a force exists. This more general approach is usually used when dealing with three dimensional forces but can be used in the two dimensional case as well.

This more general method of finding the moment of a force uses a vector operation called the cross product of two vectors.

CROSS PRODUCT (Section 4.2)



In general, the cross product of two vectors ***A*** and ***B*** results in another vector, ***C*** , i.e., ***C*** = ***A*** × ***B***. The magnitude and direction of the resulting vector can be written as

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$$

$$C = A\,B \sin\theta$$

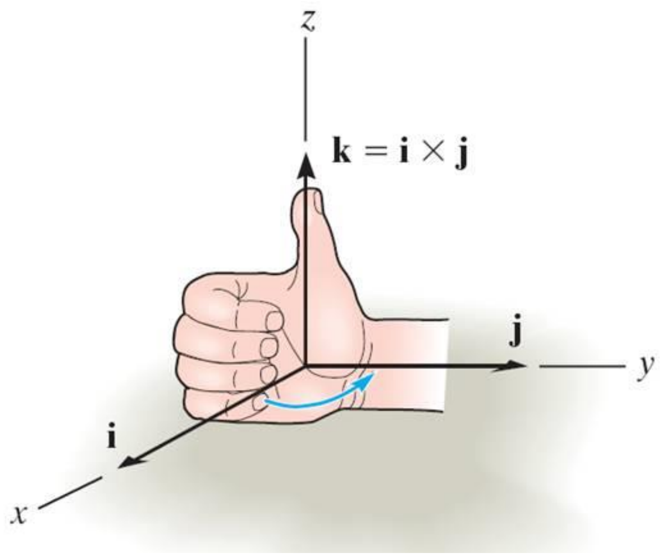
Vector C is perpendicular to both A and B vectors (or to the plane containing the A and B vectors).

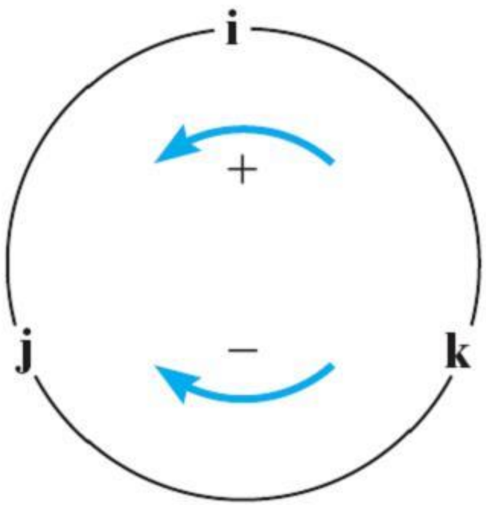
CROSS PRODUCT (continued)

The right-hand rule is a useful tool for determining the direction of the vector resulting from a cross product.

For example: $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$

Note that a vector crossed into itself is zero, e.g., $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0}$





CROSS PRODUCT (continued)

Also, the cross product can be written as a determinant.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Each component can be determined using 2 × 2 determinants.

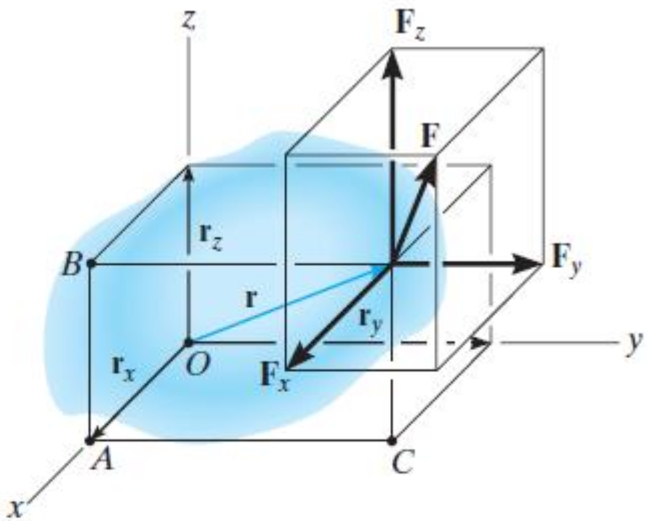
For element **i**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z - A_zB_y)$

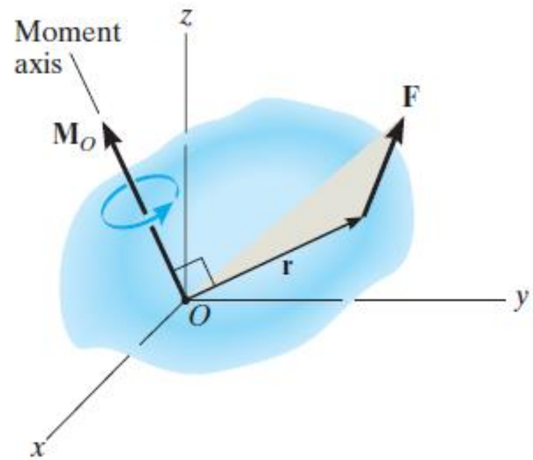
Remember the negative sign

For element **j**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_xB_z - A_zB_x)$

For element **k**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_xB_y - A_yB_x)$

MOMENT OF A FORCE – VECTOR FORMULATION (Section 4.3)





Moments in 3-D can be calculated using scalar (2-D) approach, but it can be difficult and time consuming. Thus, it is often easier to use a mathematical approach called the vector cross product.

Using the vector cross product, $\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$.

Here $\boldsymbol{r}$ is the position vector from point O to any point on the line of action of $\boldsymbol{F}$.

MOMENT OF A FORCE – VECTOR FORMULATION (continued)

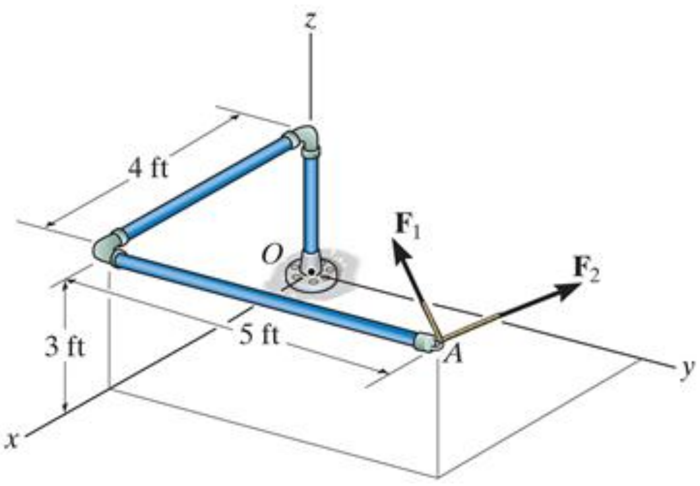
So, using the cross product, a moment can be expressed as

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

By expanding the above equation using 2 × 2 determinants, we get:

$$\boldsymbol{M_O} = (r_y\ F_z - r_z\ F_y)\ \boldsymbol{i} - (r_x F_z - r_z F_x)\ \boldsymbol{j} + (r_x F_y - r_y F_x)\ \boldsymbol{k}$$

EXAMPLE II



Given: $\boldsymbol{F_1}$={100 $\boldsymbol{i}$ - 120 $\boldsymbol{j}$ + 75 $\boldsymbol{k}$}lb

$\boldsymbol{F_2}$={-200 $\boldsymbol{i}$ +250 $\boldsymbol{j}$ + 100 $\boldsymbol{k}$}lb

Find: Resultant moment by the forces about point O.

Plan:

1) Find $\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2}$ and $\boldsymbol{r_{OA}}$.

2) Determine $\boldsymbol{M_O} = \boldsymbol{r_{OA}} \times \boldsymbol{F}$.

EXAMPLE II (continued)

Solution:

First, find the resultant force vector $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

$$= \{ (100 - 200)\ \boldsymbol{i} + (-120 + 250)\ \boldsymbol{j} + (75 + 100)\ \boldsymbol{k}\}\ \text{lb}$$

$$= \{-100\ \boldsymbol{i} + 130\ \boldsymbol{j} + 175\ \boldsymbol{k}\}\ \text{lb}$$

Find the position vector $\boldsymbol{r_{OA}}$

$$\boldsymbol{r_{OA}} = \{4\ \boldsymbol{i} + 5\ \boldsymbol{j} + 3\ \boldsymbol{k}\}\ \text{ft}$$

Then find the moment by using the vector cross product.

$$\boldsymbol{M_O} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 5 & 3 \\ -100 & 130 & 175 \end{vmatrix} = [\{5(175) - 3(130)\}\ \boldsymbol{i} - \{4(175) - 3(-100)\}\ \boldsymbol{j} + \{4(130) - 5(-100)\}\ \boldsymbol{k}]\ \text{ft}\cdot\text{lb}$$

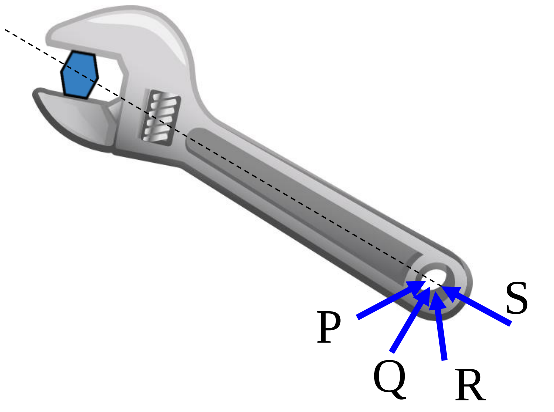
$$= \{\underline{485}\ \boldsymbol{i}\ \underline{- 1000}\ \boldsymbol{j} + \underline{1020}\ \boldsymbol{k}\}\ \underline{\text{ft}\cdot\text{lb}}$$

CONCEPT QUIZ

1. If a force of magnitude F can be applied in four different 2-D configurations (P,Q,R, & S), select the cases resulting in the maximum and minimum torque values on the nut. (Max, Min).

A) (Q, P) B) (R, S)

C) (P, R) D) (Q, S)

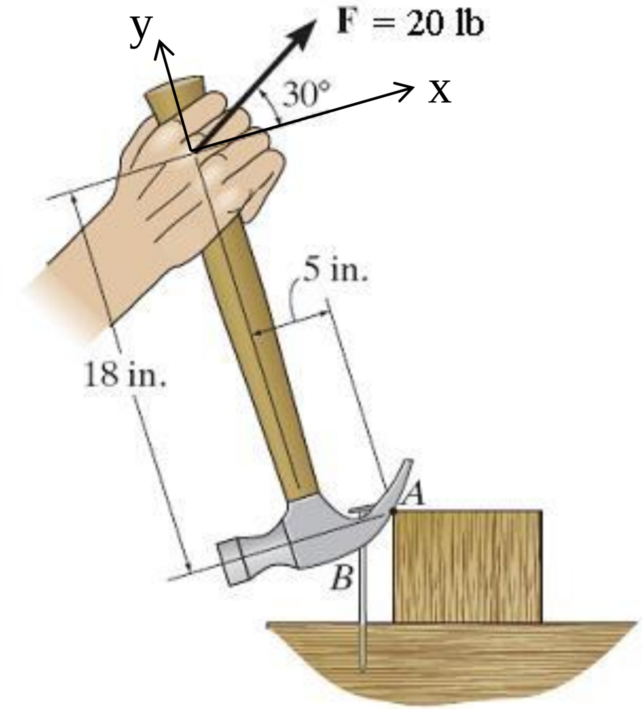


2. If $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$, then what will be the value of $\boldsymbol{M} \bullet \boldsymbol{r}$?

A) 0 B) 1

C) rF D) None of the above.

EXAMPLE III



Given: A 20 lb force is applied to the hammer.

Find: The moment of the force at A.

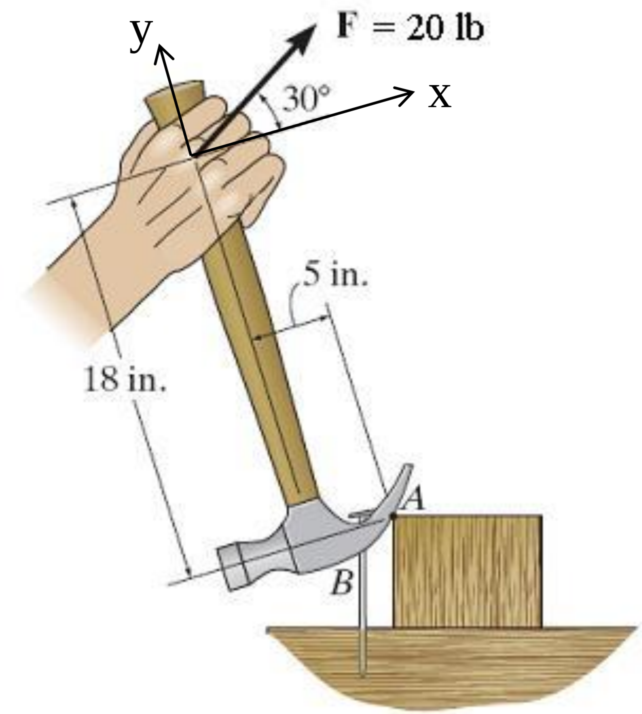
Plan:

Since this is a 2-D problem:

1) Resolve the 20 lb force along the handle's x and y axes.

2) Determine M_A using a scalar analysis.

EXAMPLE III (continued)



Solution:

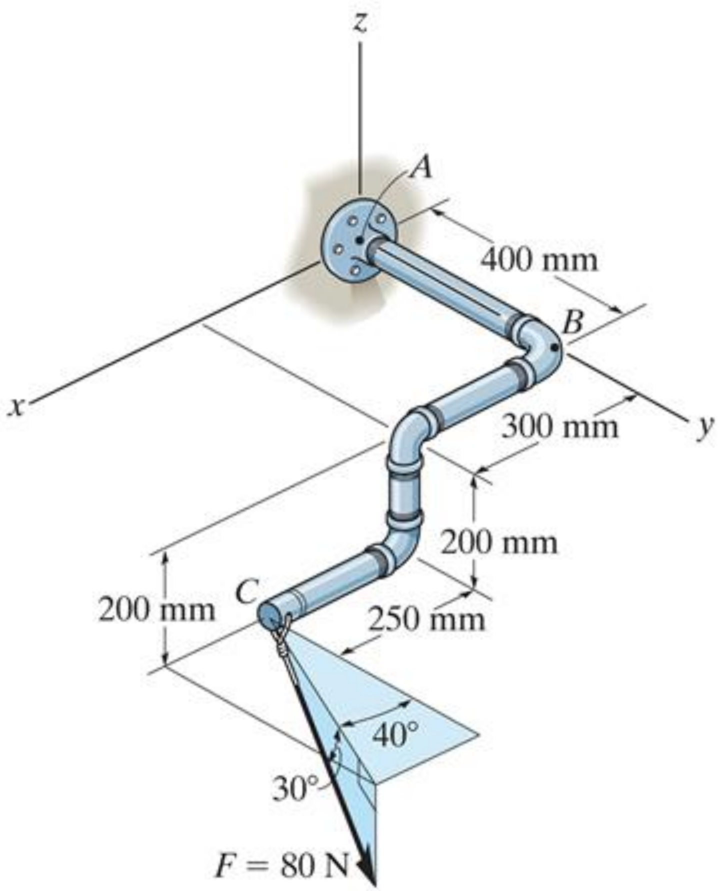
$+\uparrow F_y = 20 \sin 30° \text{ lb}$

$+\rightarrow F_x = 20 \cos 30° \text{ lb}$

$+\circlearrowleft M_A = \{-(20 \cos 30°)\text{lb } (18 \text{ in}) - (20 \sin 30°)\text{lb } (5 \text{ in})\}$

$= -361.77 \text{ lb}\cdot\text{in} =$ $\underline{362 \text{ lb}\cdot\text{in (clockwise or CW)}}$

EXAMPLE IV



Given: The force and geometry shown.

Find: Moment of F about point A

Plan:

1) Find $\boldsymbol{F}$ and $\boldsymbol{r}_{AC}$.

2) Determine $\boldsymbol{M}_A = \boldsymbol{r}_{AC} \times \boldsymbol{F}$

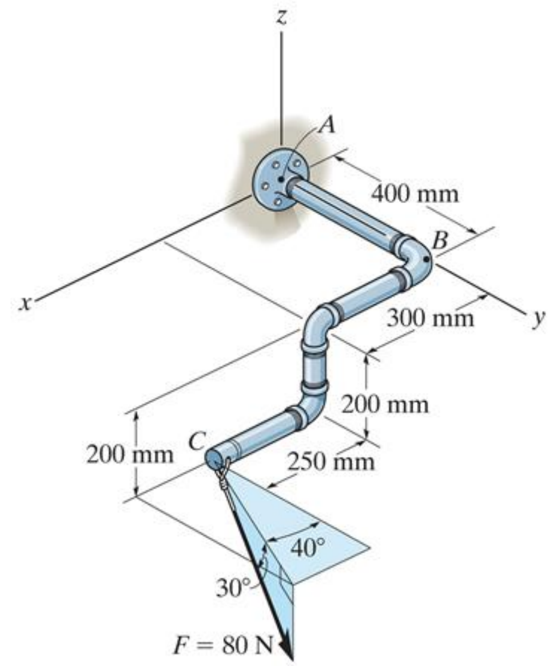
EXAMPLE IV (continued)

Solution:

$$\boldsymbol{F} = \{ (80 \cos 30) \sin 40 \, \boldsymbol{i} + (80 \cos 30) \cos 40 \, \boldsymbol{j} - 80 \sin 30 \, \boldsymbol{k} \} \text{ N}$$

$$= \{ 44.53 \, \boldsymbol{i} + 53.07 \, \boldsymbol{j} - 40 \, \boldsymbol{k} \} \text{ N}$$

$$\boldsymbol{r}_{AC} = \{ 0.55 \, \boldsymbol{i} + 0.4 \, \boldsymbol{j} - 0.2 \, \boldsymbol{k} \} \text{ m}$$

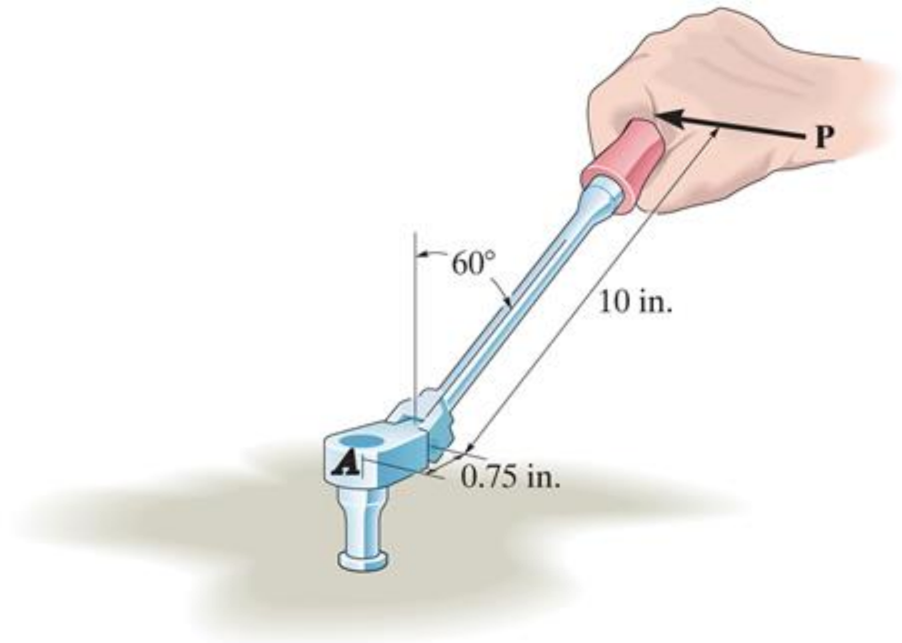


Find the moment by using the cross product.

$$\boldsymbol{M}_A = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.55 & 0.4 & -0.2 \\ 44.53 & 53.07 & -40 \end{vmatrix}$$

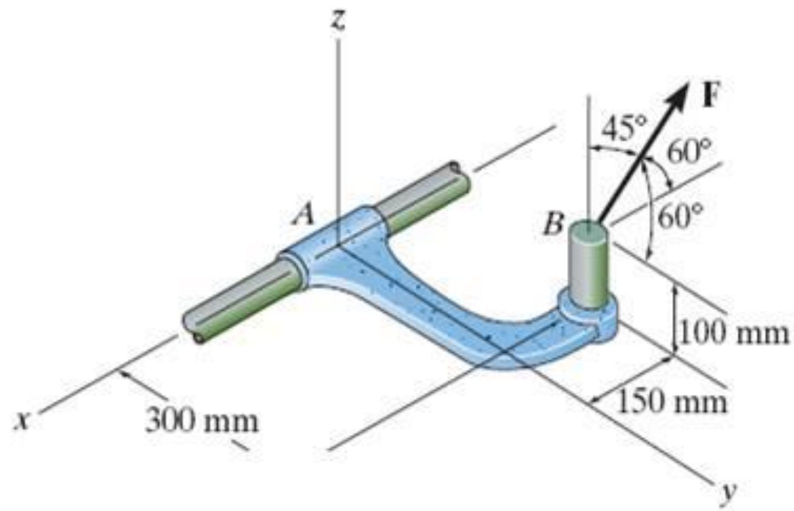
$$= \{ -5.39 \, \boldsymbol{i} + 13.1 \, \boldsymbol{j} + 11.4 \, \boldsymbol{k} \} \text{ N·m}$$

MOMENT ABOUT AN AXIS



With the force ***P***, a person is creating a moment $\boldsymbol{M_A}$ using this flex-handle socket wrench. Does all of $\boldsymbol{M_A}$ act to turn the socket? How would you calculate an answer to this question?

APPLICATION



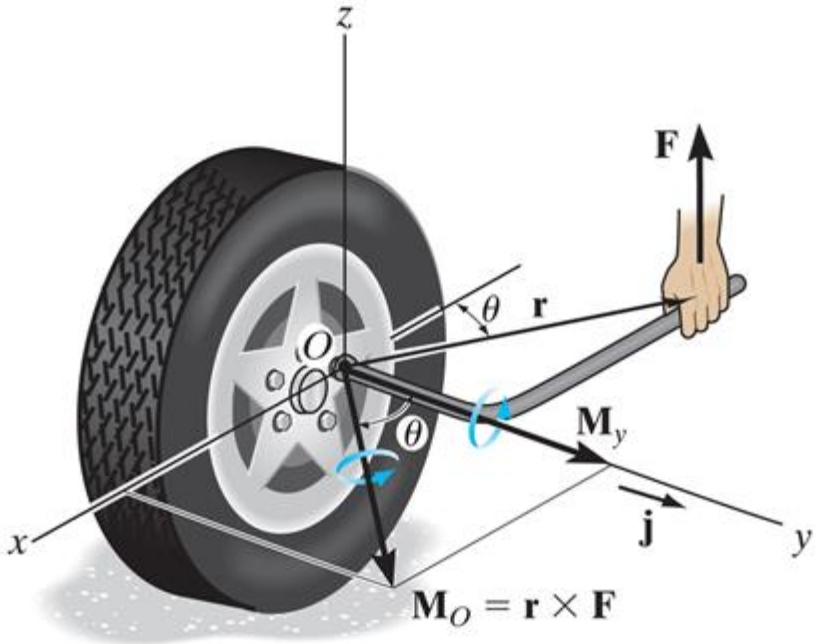
Sleeve A of this bracket can rotate about the x-axis. How would you determine the magnitude of the moment, causing by ***F*** about the x-axis?

SCALAR ANALYSIS

Recall that the moment of a scalar force about any point O is $M_O = F\, d_O$ where d_O is the perpendicular (or shortest) distance from the point to the force's line of action. This concept can be extended to find the moment of a force about an axis.

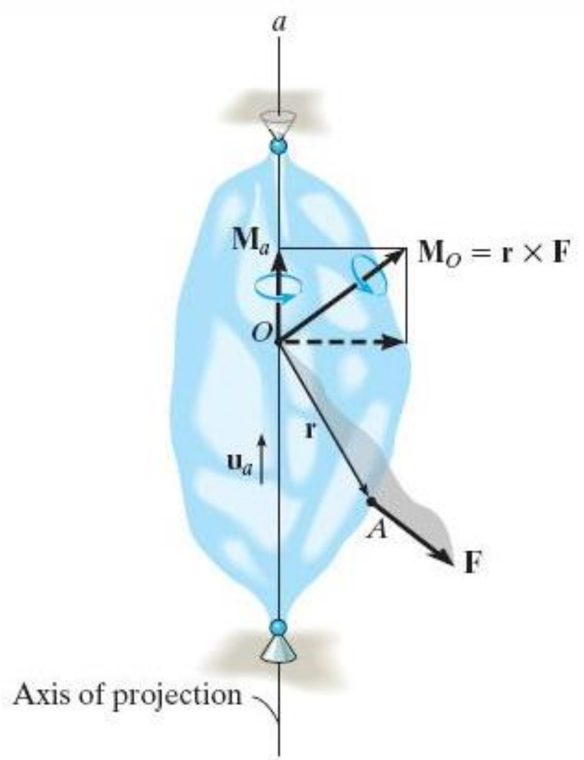
Finding the moment of a force about an axis can help answer the types of questions we just considered.

SCALAR ANALYSIS (continued)



In the figure above, the moment about the y-axis would be $M_y = F_z\ (d_x) = F\ (r \cos \theta)$. However, unless the force can easily be broken into components and the "d_x" found quickly, such calculations are not always trivial and vector analysis may be much easier (and less likely to produce errors).

VECTOR ANALYSIS



Our goal is to find the moment of ***F*** (the tendency to rotate the body) about the *a*-axis.

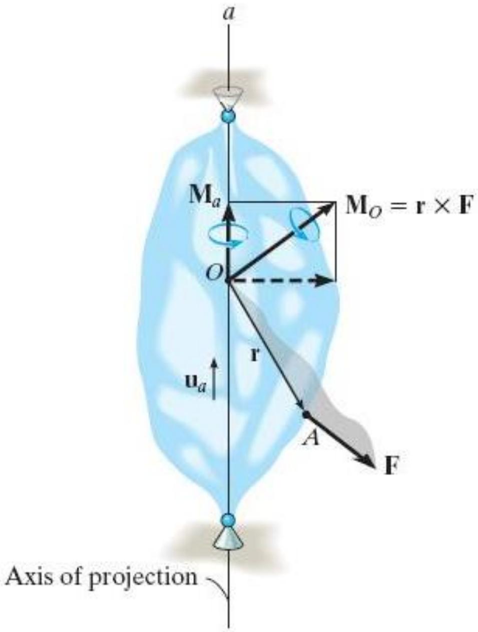
First compute the moment of ***F*** about any arbitrary point O that lies on the *a*-axis using the cross product.

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

Now, find the component of $\boldsymbol{M_O}$ along the *a*-axis using the dot product.

$$M_a = \boldsymbol{u_a} \cdot \boldsymbol{M_O}$$

VECTOR ANALYSIS (continued)



M_a can also be obtained as

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

The above equation is also called the triple scalar product.

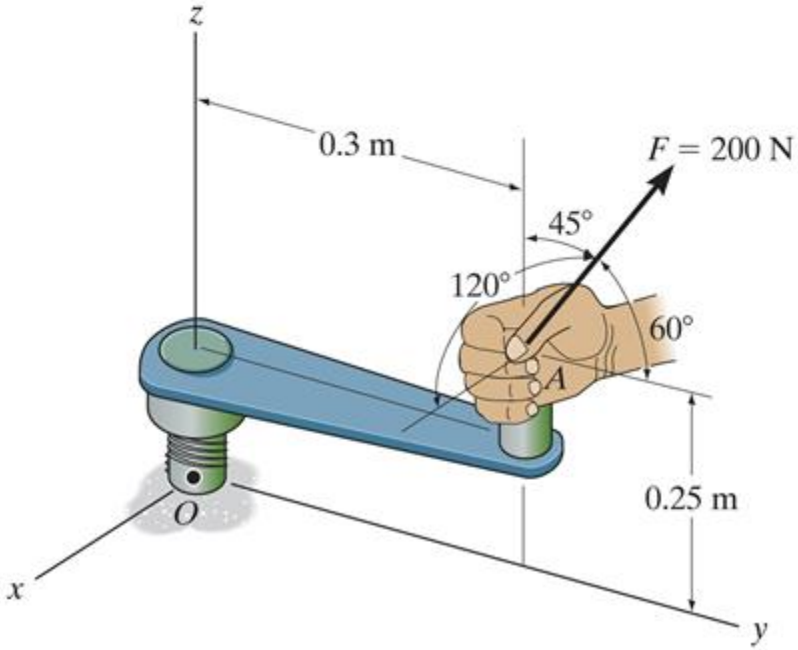
In the this equation,

$\boldsymbol{u_a}$ represents the unit vector along the *a*-axis,

$\boldsymbol{r}$ is the position vector from any point on the *a*-axis to any point A on the line of action of the force, and

$\boldsymbol{F}$ is the force vector.

EXAMPLE V



Given: A force is applied to the tool as shown.

Find: The magnitude of the moment of this force about the x axis.

Plan:

1) Use $M_x = \boldsymbol{u} \bullet (\boldsymbol{r} \times \boldsymbol{F})$.

2) First, find ***F*** in Cartesian vector form.

3) Note that ***u*** = 1 ***i*** in this case.

4) The vector ***r*** is the position vector from O to A.

EXAMPLE V (continued)

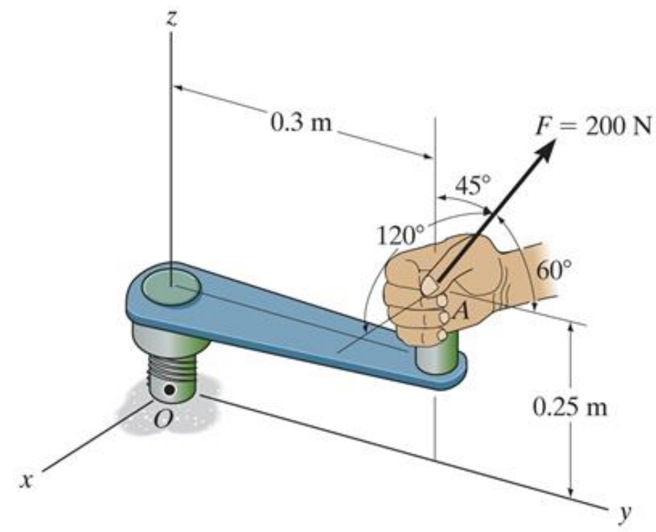
Solution:

$\boldsymbol{u} = 1\,\boldsymbol{i}$

$\boldsymbol{r}_{OA} = \{0\,\boldsymbol{i} + 0.3\,\boldsymbol{j} + 0.25\,\boldsymbol{k}\}$ m

$\boldsymbol{F} = 200\,(\cos 120\,\boldsymbol{i} + \cos 60\,\boldsymbol{j} + \cos 45\,\boldsymbol{k})$ N

$= \{-100\,\boldsymbol{i} + 100\,\boldsymbol{j} + 141.4\,\boldsymbol{k}\}$ N



Now find $M_x = \boldsymbol{u} \bullet (\boldsymbol{r}_{OA} \times \boldsymbol{F})$

$$M_x = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0.25 \\ -100 & 100 & 141.4 \end{vmatrix} = 1\{0.3\,(141.4) - 0.25\,(100)\}\ \text{N·m}$$

$\underline{M_x = 17.4\ \text{N·m CCW}}$

CONCEPT QUIZ

1. The vector operation $(\boldsymbol{P} \times \boldsymbol{Q}) \bullet \boldsymbol{R}$ equals

   A) $\boldsymbol{P} \times (\boldsymbol{Q} \bullet \boldsymbol{R})$.

   B) $\boldsymbol{R} \bullet (\boldsymbol{P} \times \boldsymbol{Q})$.

   C) $(\boldsymbol{P} \bullet \boldsymbol{R}) \times (\boldsymbol{Q} \bullet \boldsymbol{R})$.

   D) $(\boldsymbol{P} \times \boldsymbol{R}) \bullet (\boldsymbol{Q} \times \boldsymbol{R})$.

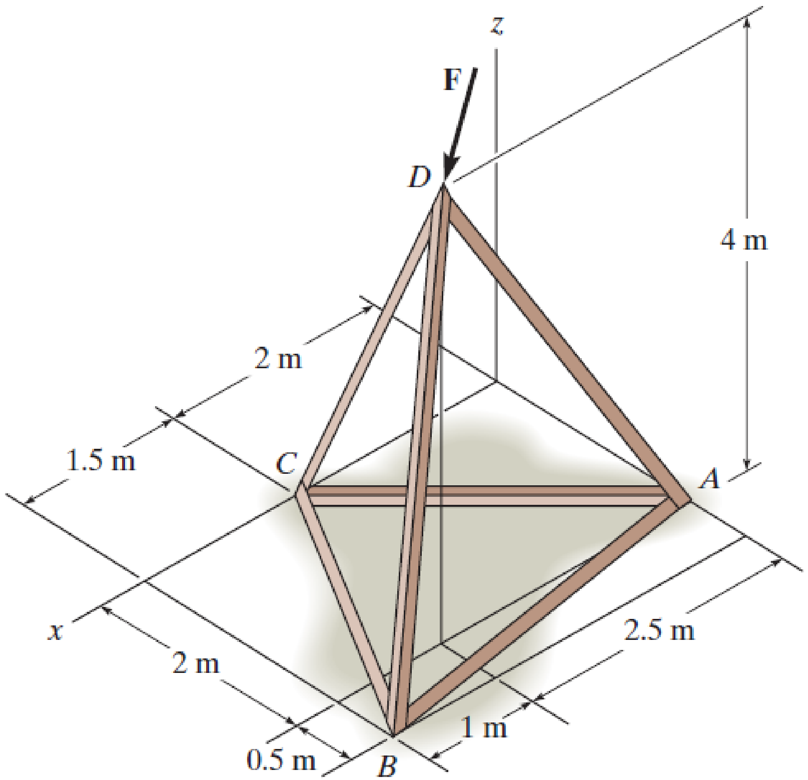
CONCEPT QUIZ (continued)

2. The force ***F*** is acting along DC. Using the triple scalar product to determine the moment of ***F*** about the bar BA, you could use any of the following position vectors except ______.

   A) $\boldsymbol{r}_{BC}$ B) $\boldsymbol{r}_{AD}$

   C) $\boldsymbol{r}_{AC}$ D) $\boldsymbol{r}_{DB}$

   E) $\boldsymbol{r}_{BD}$



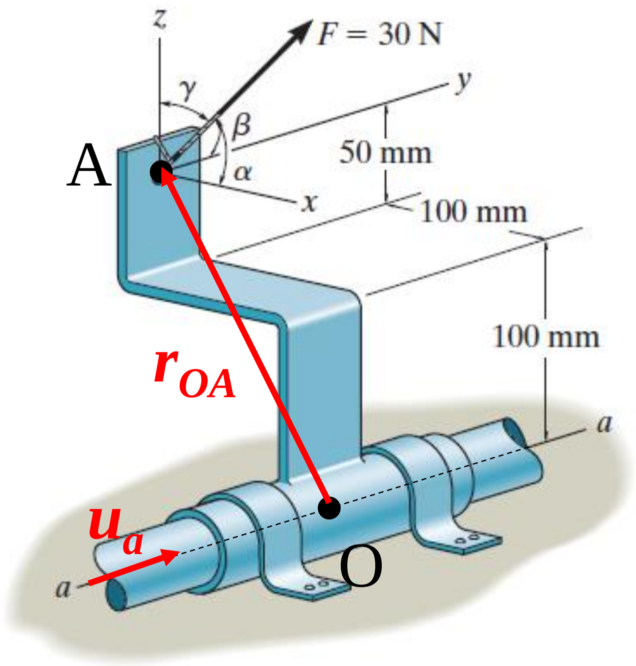
EXAMPLE VI

Given: The force of F = 30 N acts on the bracket.
$\alpha = 60°$, $\beta = 60°$, $\gamma = 45°$.

Find: The moment of ***F*** about the a-a axis.

Plan:

1) Find $\boldsymbol{u_a}$ and $\boldsymbol{r_{OA}}$
2) Find ***F*** in Cartesian vector form.
3) Use $M_a = \boldsymbol{u_a} \bullet (\boldsymbol{r_{OA}} \times \boldsymbol{F})$

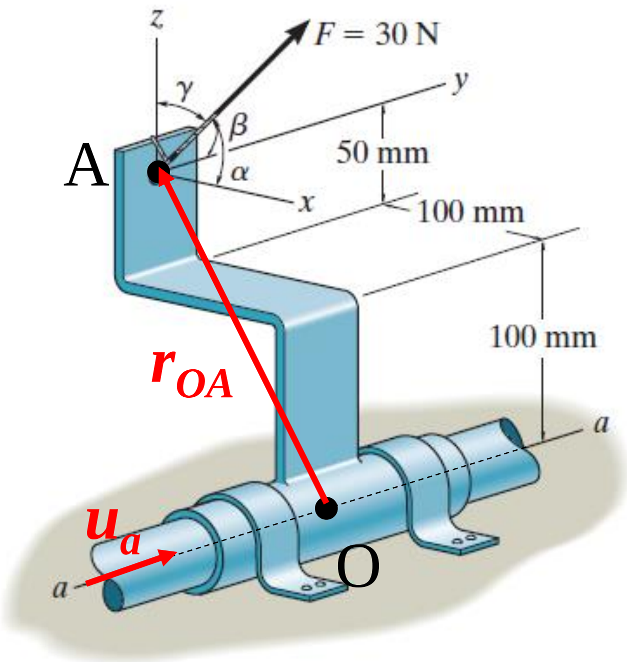


EXAMPLE VI (continued)

Solution:

$\boldsymbol{u_a} = \boldsymbol{j}$

$\boldsymbol{r_{OA}} = \{-0.1\ \boldsymbol{i} + 0.15\ \boldsymbol{k}\}$ m



$\boldsymbol{F} = 30\ \{\cos 60°\ \boldsymbol{i} + \cos 60°\ \boldsymbol{j} + \cos 45°\ \boldsymbol{k}\}$ N

$\boldsymbol{F} = \{\ 15\ \boldsymbol{i} + 15\ \boldsymbol{j} + 21.21\ \boldsymbol{k}\}$ N

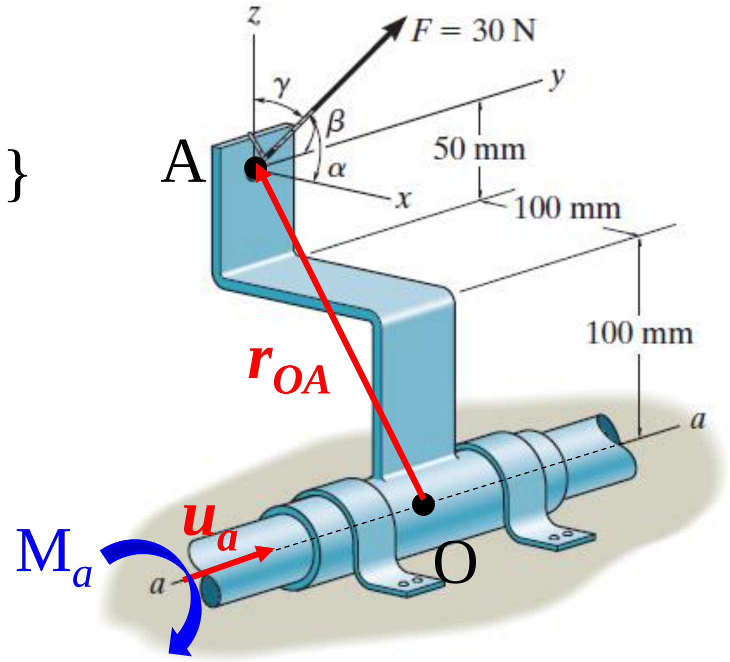
EXAMPLE VI (continued)

Now find the triple product, $M_a = \boldsymbol{u_a} \bullet (\boldsymbol{r_{OA}} \times \boldsymbol{F})$

$$M_a = \begin{vmatrix} 0 & 1 & 0 \\ -0.1 & 0 & 0.15 \\ 15 & 15 & 21.21 \end{vmatrix} \text{ N·m}$$

$$M_a = -1\,\{-0.1\,(21.21) - 0.15\,(15)\}$$

$$= \underline{4.37 \text{ N·m}}$$

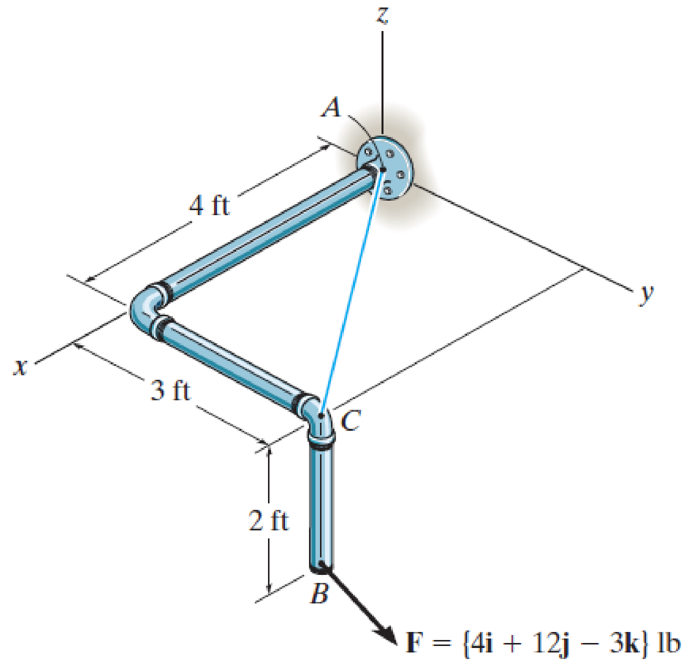


ATTENTION QUIZ

1. For finding the moment of the force ***F*** about the x-axis, the position vector in the triple scalar product should be ___ .

   A) $\boldsymbol{r}_{AC}$ B) $\boldsymbol{r}_{BA}$

   C) $\boldsymbol{r}_{AB}$ D) $\boldsymbol{r}_{BC}$



2. If $\boldsymbol{r} = \{1\ \boldsymbol{i} + 2\ \boldsymbol{j}\}$ m and $\boldsymbol{F} = \{10\ \boldsymbol{i} + 20\ \boldsymbol{j} + 30\ \boldsymbol{k}\}$ N, then the moment of ***F*** about the y-axis is _____ N·m.

   A) 10 B) -30

   C) -40 D) None of the above.