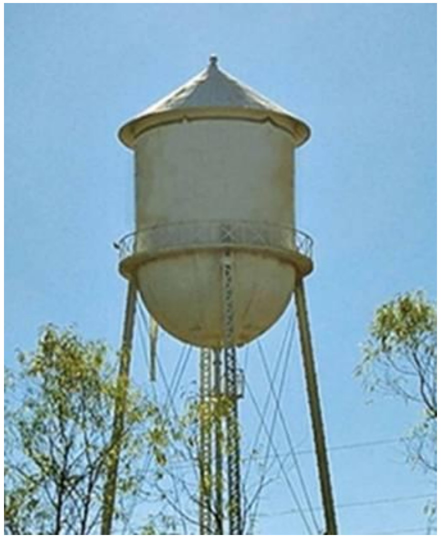
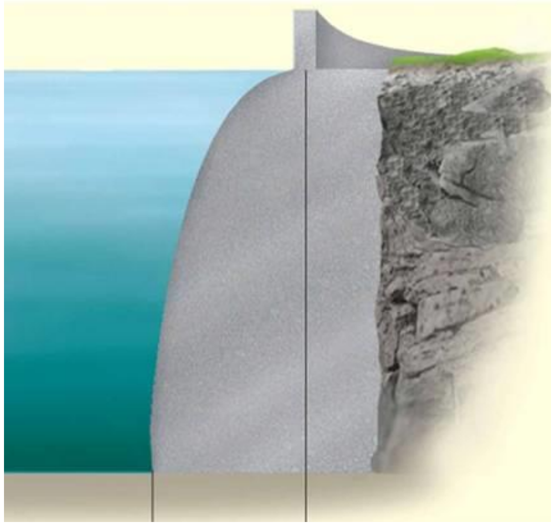
CENTER OF GRAVITY, CENTER OF MASS AND CENTROID OF A BODY

Objectives :

a) Understand the concepts of center of gravity, center of mass, and centroid.

b) Be able to determine the location of these points for a body.



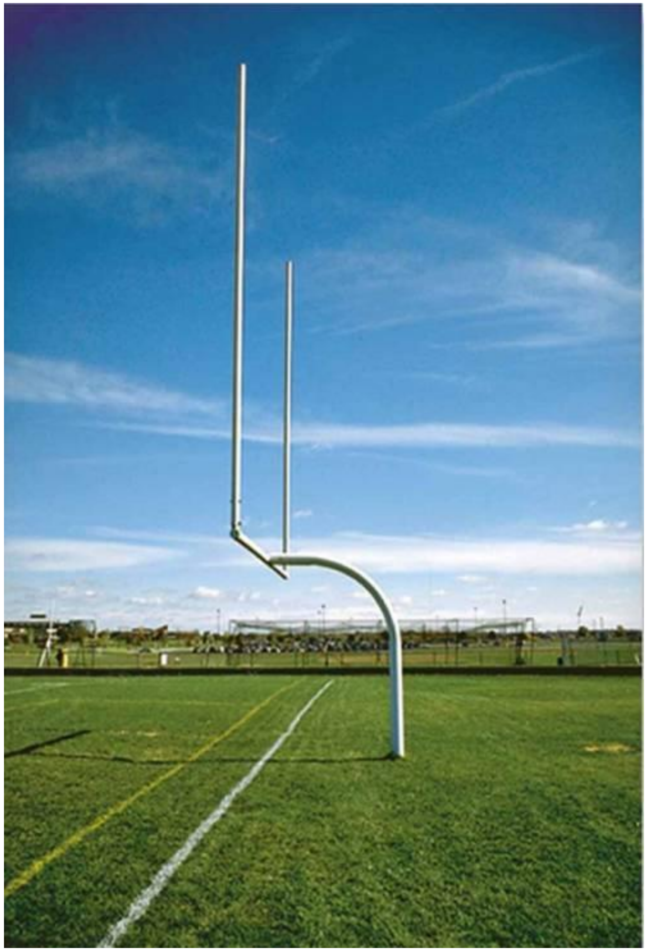


APPLICATIONS

To design the structure for supporting a water tank, we will need to know the weight of the tank and water as well as the locations where the resultant forces representing these distributed loads act.

How can we determine these resultant weights and their lines of action?

APPLICATIONS (continued)

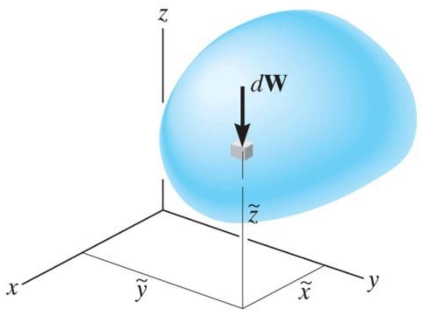


To design the ground support structure for a goal post, it is critical to find total weight of the structure and the center of gravity's location.

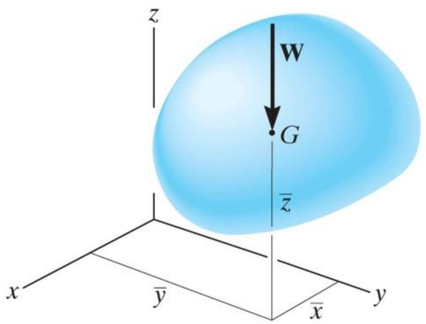
Integration must be used to determine total weight of the goal post due to the curvature of the supporting member.

How do you determine the location of overall center of gravity?

CONCEPT OF CENTER OF GRAVITY (CG)



A body is composed of an infinite number of particles, and so if the body is located within a gravitational field, then each of these particles will have a weight $d\boldsymbol{W}$.

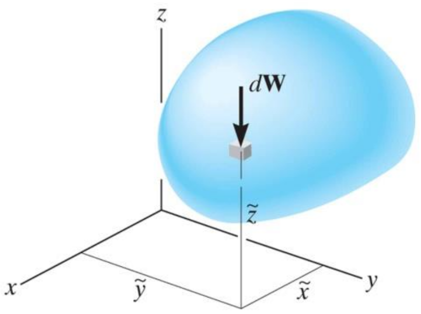


The center of gravity (CG) is a point, often shown as G, which locates the resultant weight of a system of particles or a solid body.

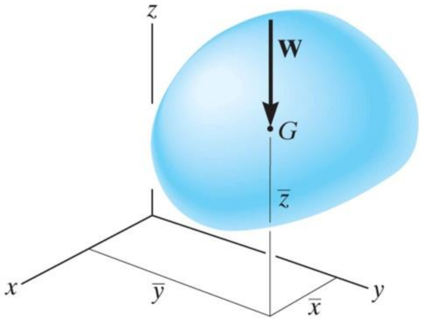
From the definition of a resultant force, the sum of moments due to individual particle weight about any point is the same as the moment due to the resultant weight located at G.

Also, note that the sum of moments due to the individual particle's weights about point G is equal to zero.

CONCEPT OF CG (continued)



The location of the center of gravity, measured from the y axis, is determined by equating the moment of ***W*** about the y-axis to the sum of the moments of the weights of the particles about this same axis.



If $d\mathbf{W}$ is located at point $(\tilde{x}, \tilde{y}, \tilde{z})$, then

$$\bar{x}\, \mathbf{W} = \int \tilde{x}\, d\mathbf{W}$$

Similarly, $\bar{y}\, \mathbf{W} = \int \tilde{y}\, d\mathbf{W}$

Location of the center of gravity G with respect to the x, y, z-axes becomes

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$

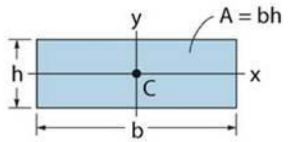
CM & CENTROID OF A BODY

By replacing the ***W*** with m in these equations, the coordinates of the center of mass can be found.

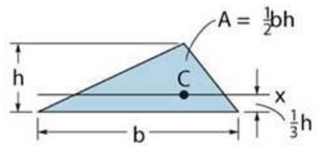
$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\, dm}{\int dm}$$

Similarly, the coordinates of the centroid of volume, area, or length can be obtained by replacing ***W*** by ***V***, ***A***, or ***L***, respectively.

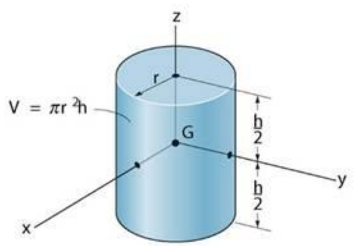
CONCEPT OF CENTROID



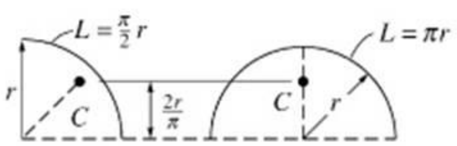
Rectangular area



Triangular area



Cylinder



Quarter and semicircle arcs

The centroid, C, is a point defining the geometric center of an object.

The centroid coincides with the center of mass or the center of gravity only if the material of the body is homogenous (density or specific weight is constant throughout the body).

If an object has an axis of symmetry, then the centroid of object lies on that axis.

In some cases, the centroid may not be located on the object.

STEPS TO DETERME THE CENTROID OF AN AREA

1. Choose an appropriate differential element dA at a general point (x,y). Hint: Generally, if y is easily expressed in terms of x (e.g., $y = x^2 + 1$), use a vertical rectangular element. If the converse is true, then use a horizontal rectangular element.

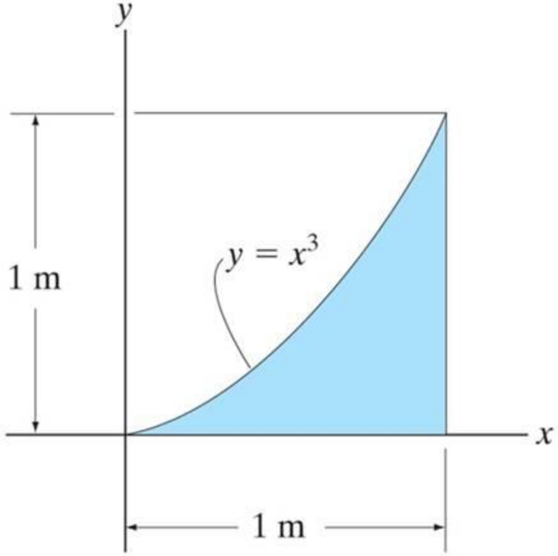
2. Express dA in terms of the differentiating element dx (or dy).

3. Determine coordinates ($\tilde{x}$, $\tilde{y}$) of the centroid of the rectangular element in terms of the general point (x, y).

4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

Note: Similar steps are used for determining the CG or CM. These steps will become clearer by doing a few examples.

EXAMPLE I

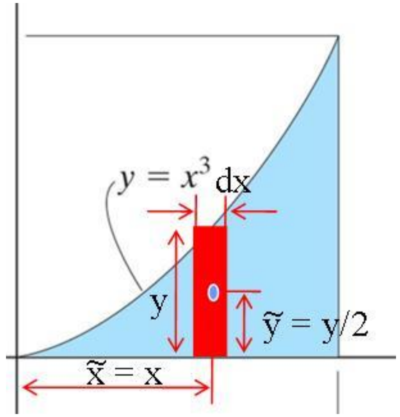


Given: The area as shown.

Find: The centroid location ($\bar{x}$, $\bar{y}$)

Plan: Follow the steps.

Solution:



1. Since y is given in terms of x, choose dA as a vertical rectangular strip.

2. $dA = y\,dx = x^3\,dx$

3. $\tilde{x} = x$ and $\tilde{y} = y / 2 = x^3 / 2$

EXAMPLE I (continued)

4. $\bar{x} = \left(\int_A \tilde{x} \; dA \right) / \left(\int_A dA \right)$

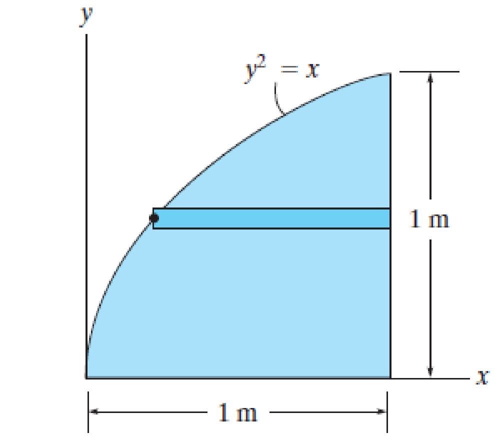
$$= \frac{\int_0^1 x\,(x^3)\,d\,x}{\int_0^1 (x^3)\,d\,x} = \frac{1/5\,[\,x^5\,]_0^1}{1/4\,[\,x^4\,]_0^1}$$

$= (1/5) / (1/4) = 0.8 \text{ m}$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^1 (x^3/2)\,(x^3)\,dx}{\int_0^1 x^3\,dx} = \frac{1/14[x^7]_0^1}{1/4}$$

$= (1/14) / (1/4) = 0.2857 \text{ m}$

EXAMPLE II

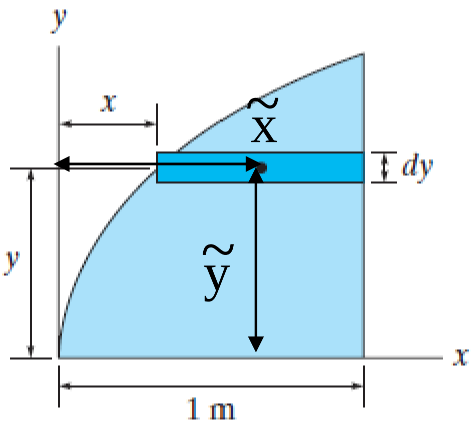


Given: The shape and associated horizontal rectangular strip shown.

Find: dA and $(\tilde{x}, \tilde{y})$

Plan: Follow the steps.

Solution:

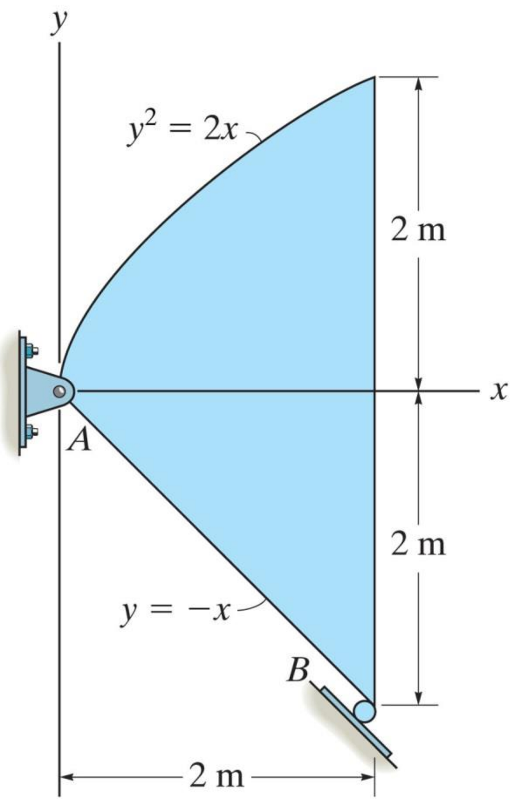


1. $dA = (1\text{-}x)\ dy = \underline{(1\text{-}y^2)\ dy}$

2. $\tilde{x} = x + (1–x) / 2 = (1 + x) /2 = \underline{(1 + y^2)/2}$

3. $\tilde{y} = \overline{y}$

EXAMPLE III



Given: The steel plate is 0.3 m thick and has a density of 7850 kg/m^3.

Find: The location of its center of mass.

Plan:

Follow the solution steps to find the CM by integration.

EXAMPLE III (continued)

Solution:

1. Choose dA as a vertical rectangular strip.

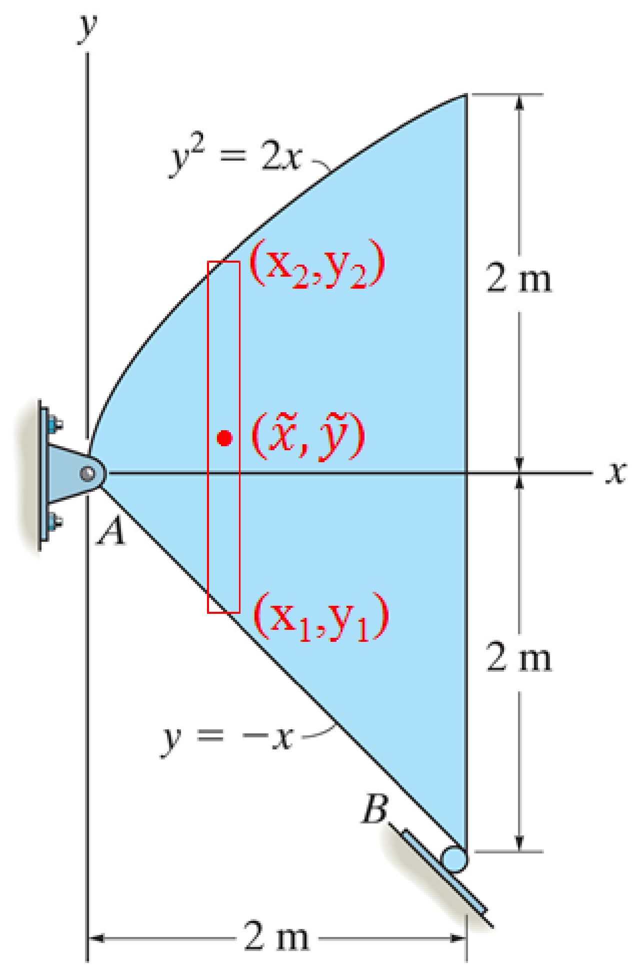
2. $dA = (y_2 - y_1)\ dx$

   $= (\sqrt{2x} + x)\ dx$

3. $\tilde{x} = x$

   $\tilde{y} = (y_1 + y_2) / 2$

   $= (\sqrt{2x} - x) / 2$



EXAMPLE III (continued)

4.

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^2 x\left(\sqrt{2x}+x\right)dx}{\int_0^2\left(\sqrt{2x}+x\right)dx} = \frac{\left[\left(\frac{2\sqrt{2}}{5}\right)x^{\frac{5}{2}}+\frac{1}{3}x^3\right]_0^2}{\left[\left(\frac{2\sqrt{2}}{3}\right)x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_0^2}$$

$$= \frac{5.867}{4.667} = 1.257\text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^2\{(\sqrt{2x}-x)/2\}\left(\sqrt{2x}+x\right)dx}{\int_0^2\left(\sqrt{2x}+x\right)dx} = \frac{\left[\frac{x^2}{2}-\frac{1}{6}x^3\right]_0^2}{\left[\left(\frac{2\sqrt{2}}{3}\right)x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_0^2}$$

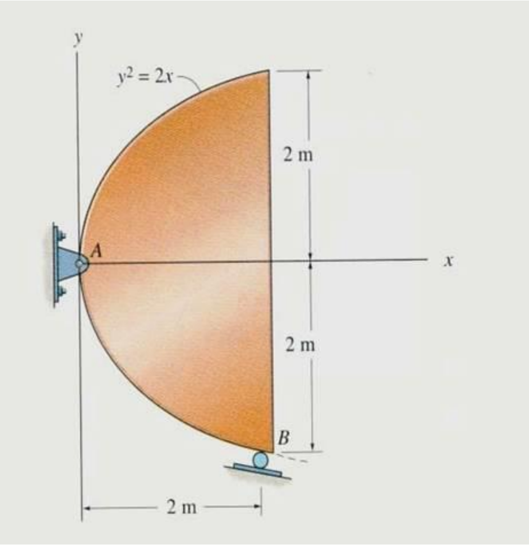
$$= \frac{0.6667}{4.667} = 0.143\text{ m}$$

$\bar{x}$ = 1.26 m and $\bar{y}$ = 0.143 m

CONCEPT QUIZ

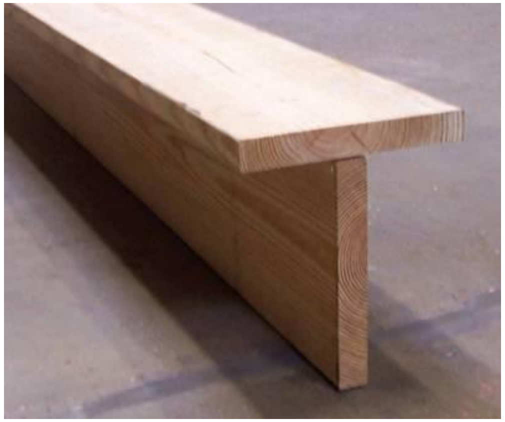
1. The steel plate, with known weight and non-uniform thickness and density, is supported as shown. Of the parameters CG/CM and centroid, which one is needed for determining the support reactions? Are both parameters located at the same point?

   A) (center of gravity, yes)
   B) (center of gravity, no)
   C) (centroid, yes)
   D) (centroid, no)



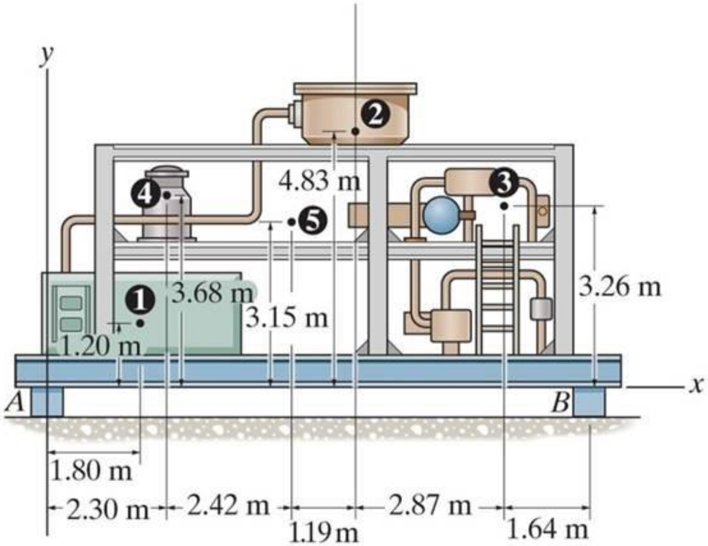
2. When determining the centroid of the area above, which type of differential area element requires the least computational work, Vertical or Horizontal?

COMPOSITE BODIES



- A composite body in this section refers to a body made of a collection of “simple” shaped parts or holes.
- The I-beam (top) or T-beam (bottom) shown are commonly used in building various types of structures. When doing a stress or deflection analysis for a beam, the location of its centroid is very important.
- How can we easily determine the location of the centroid for different beam shapes?

APPLICATION



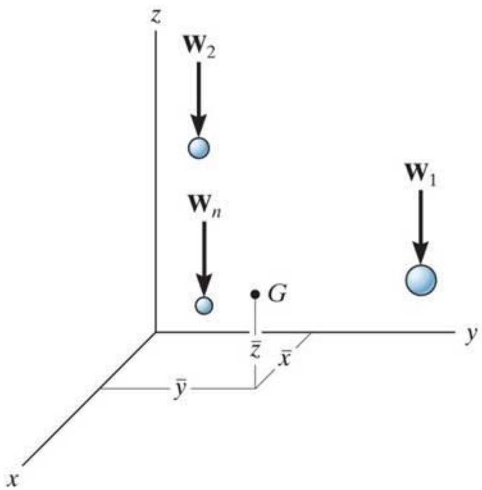
1	Instrument panel	230 kg
2	Filter system	183 kg
3	Piping assembly	120 kg
4	Liquid storage	85 kg
5	Structural framework	468 kg

The compressor is assembled with many individual components.

In order to design the ground support structures, the reactions at blocks A and B have to be found. To do this easily, it is important to determine the location of the compressor's center of gravity (CG).

If we know the weight and CG of individual components, we need a simple way to determine the location of the CG of the assembled unit.

CG/CM OF A COMPOSITE BODY



Consider a composite body which consists of a series of particles (or bodies) as shown in the figure. The net or resultant weight is given as $W_R = \sum W$.

Summing the moments about the y-axis, we get

$$\bar{x}\, W_R = \tilde{x}_1 W_1 + \tilde{x}_2 W_2 + \ldots + \tilde{x}_n W_n$$

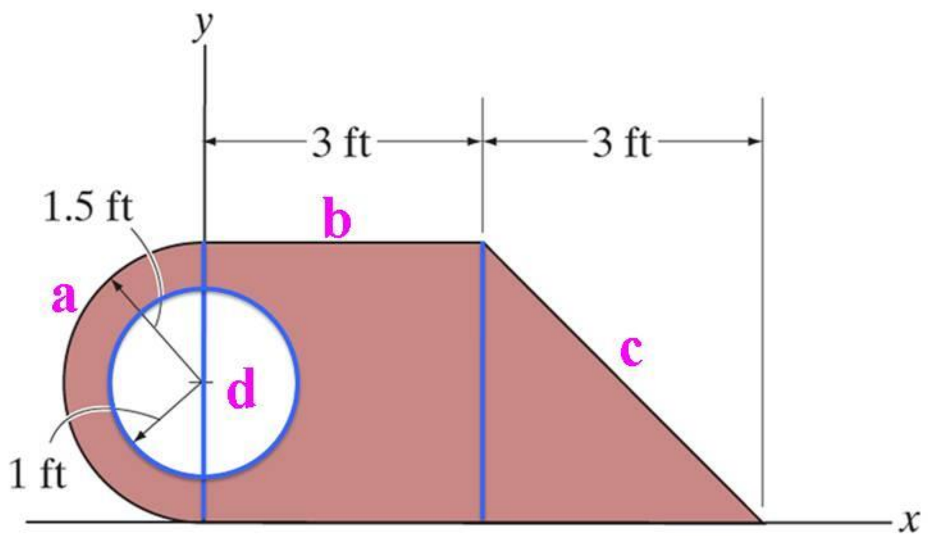
where $\tilde{x}_1$ represents x coordinate of W_1, etc..

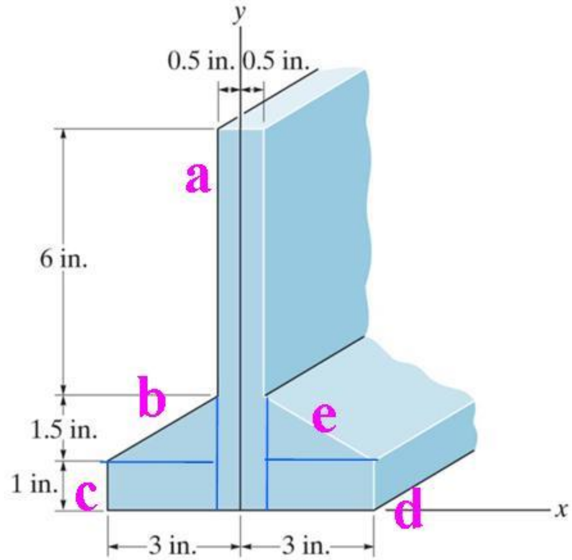
Similarly, we can sum moments about the x- and z-axes to find the coordinates of the CG.

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}$$

By replacing the W with a m in these equations, the coordinates of the center of mass can be found.

CONCEPT OF A COMPOSITE BODY

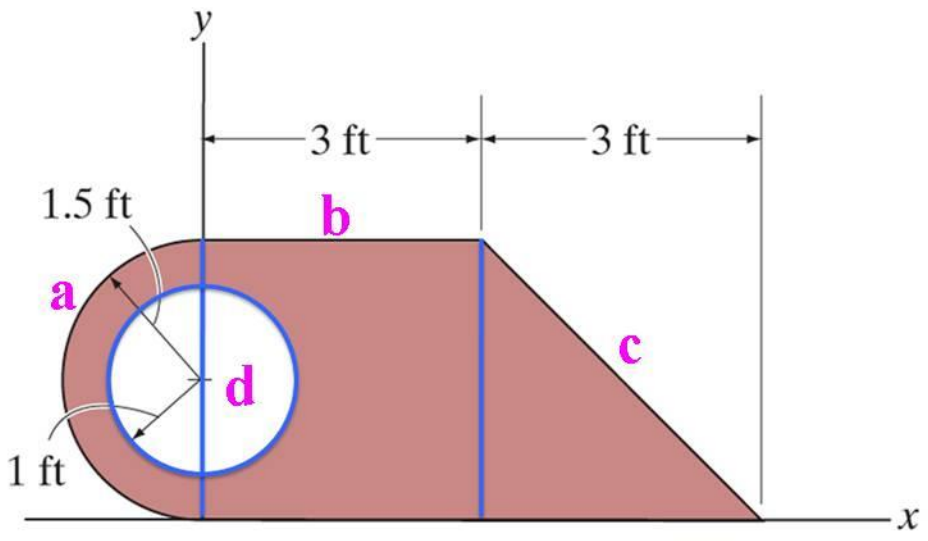


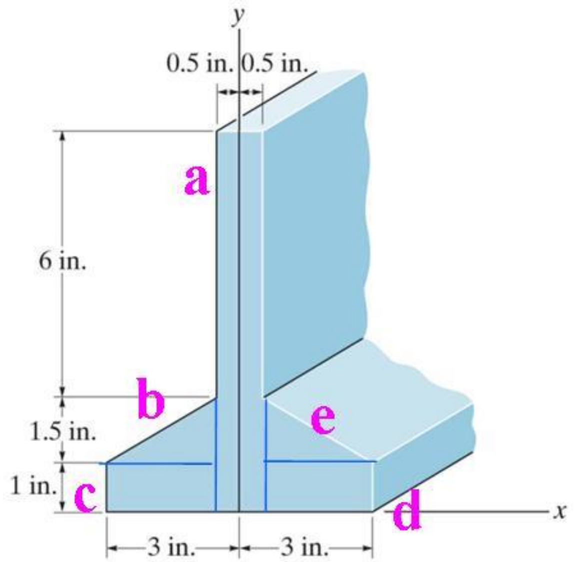


Many industrial objects can be considered as composite bodies made up of a series of connected "simple-shaped" parts, like a rectangle, triangle, and semicircle, or holes.

Knowing the location of the centroid, C, or center of gravity, CG, of the simple-shaped parts, we can easily determine the location of the C or CG for the more complex composite body.

CONCEPT OF A COMPOSITE BODY (continued)





This can be done by considering each part as a "particle" and following the procedure as described in Section 9.1.

This is a simple, effective, and practical method of determining the location of the centroid or center of gravity of a complex part, structure or machine.

STEPS FOR ANALYSIS

1. Divide the body into pieces that are known shapes.
   Holes are considered as pieces with negative weight or size.

2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.

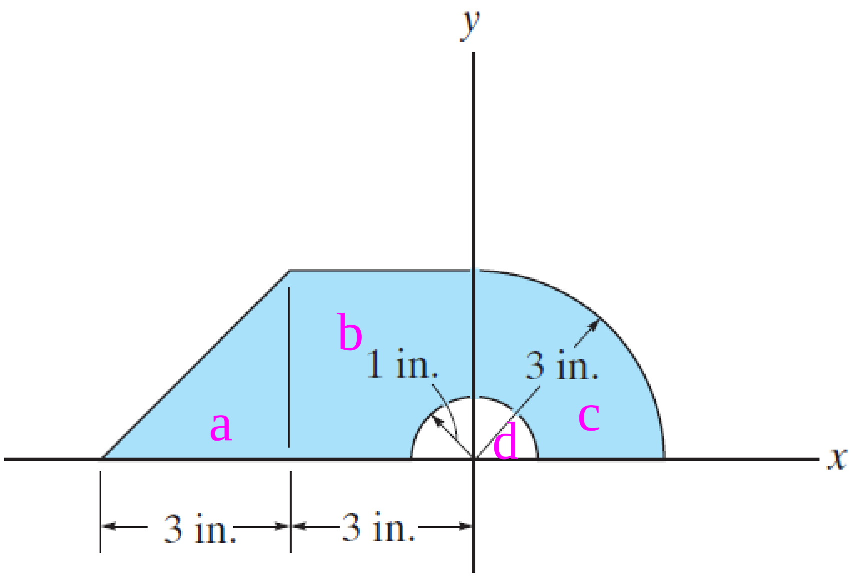
3. Fix the coordinate axes, determine the coordinates of the center of gravity of centroid of each piece, and then fill in the table.

4. Sum the columns to get $\bar{x}$, $\bar{y}$, and $\bar{z}$. Use formulas like

   $$\bar{x} = (\Sigma \tilde{x}_i A_i) / (\Sigma A_i) \quad \text{or} \quad \bar{x} = (\Sigma \tilde{x}_i W_i) / (\Sigma W_i)$$

   This approach will become straightforward after doing examples!

EXAMPLE IV



Given: The part shown.

Find: The centroid of the part.

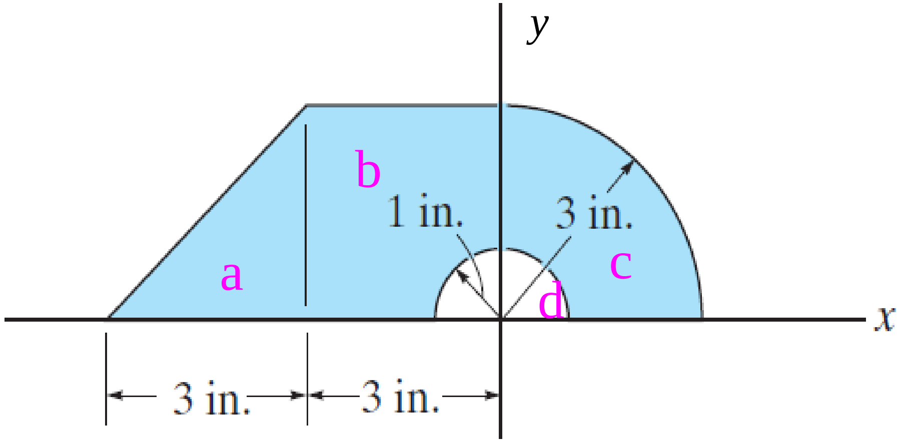
Plan: Follow the steps for analysis.

Solution:

1. This body can be divided into the following pieces:
   triangle (a) + rectangle (b) + quarter circular (c)
   – semicircular area (d).
   Note that a negative sign should be used for the hole!

EXAMPLE IV (continued)

Steps 2 & 3: Create and complete the table using parts a, b, c, and d. Note the location of the axis system.



Segment	Area A (in^2)	$\tilde{x}$ (in)	$\tilde{y}$ (in)	$\tilde{x}$ A (in^3)	$\tilde{y}$ A (in^3)
Triangle a	4.5	– 4	1	– 18	4.5
Rectangle b	9.0	– 1.5	1.5	– 13.5	13.5
Qtr. Circle c	$9 \pi / 4$	$4(3) / (3 \pi)$	$4(3) / (3 \pi)$	9	9
Semi-Circle d	$– \pi / 2$	0	$4(1) / (3 \pi)$	0	– 0.67
Σ	**19.00**			**– 22.5**	**26.33**

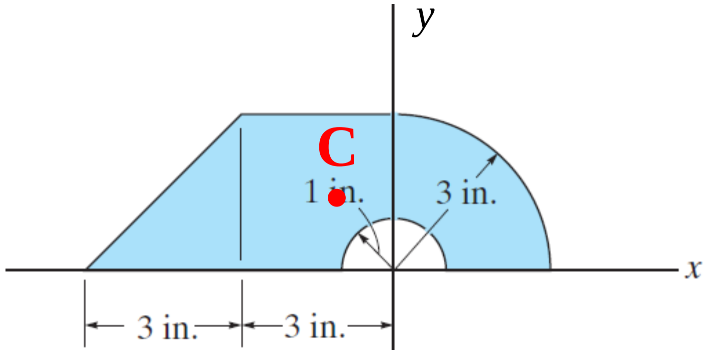
EXAMPLE IV (continued)

4. Now use the table data results and the formulas to find the coordinates of the centroid.

Area A	$\tilde{x}$ A	$\tilde{y}$ A
19.00	– 22.5	26.33

$$\bar{x} = (\Sigma\, \tilde{x}\, A) / (\Sigma\, A) = -\,22.5\ \text{in}^3 / 19.0\ \text{in}^2 = \underline{-\,1.18\ \text{in}}$$

$$\bar{y} = (\Sigma\, \tilde{y}\, A) / (\Sigma\, A) = 26.33\ \text{in}^3 / 19.0\ \text{in}^2 = \underline{1.39\ \text{in}}$$

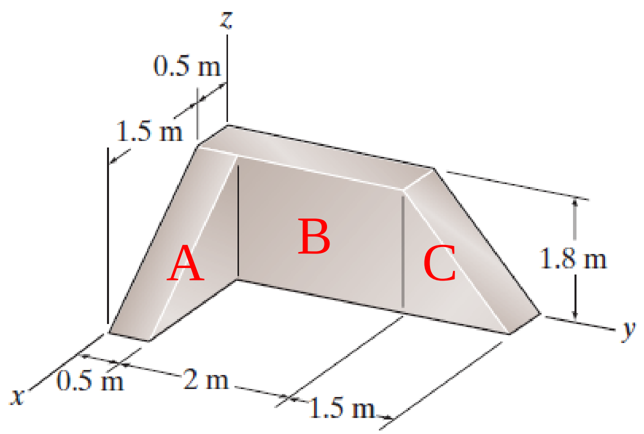


EXAMPLE V

Given: Three blocks are assembled as shown.

Find: The center of volume of this assembly.

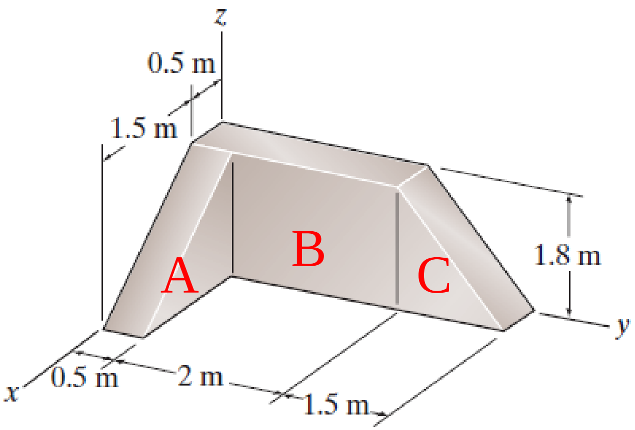
Plan: Follow the steps for analysis.



Solution:

1. In this problem, the blocks A, B and C can be considered as three pieces (or segments).

EXAMPLE V (continued)



Volumes of each shape:

$V_A = (0.5)\,(1.5)\,(1.8)\,(0.5) = 0.675\ m^3$

$V_B = (2.5)\,(1.8)\,(0.5) = 2.25\ m^3$

$V_C = (0.5)\,(1.5)\,(1.8)\,(0.5) = 0.675\ m^3$

Segment	V (m^3)	$\tilde{x}$ (m)	$\tilde{y}$ (m)	$\tilde{z}$ (m)	$\tilde{x}V$ (m^4)	$\tilde{y}V$ (m^4)	$\tilde{z}V$ (m^4)
A	0.675	1.0	0.25	0.6	0.675	0.1688	0.405
B	2.25	0.25	1.25	0.9	0.5625	2.813	2.025
C	0.675	0.25	3.0	0.6	0.1688	2.025	0.405
Σ	**3.6**				**1.406**	**5.007**	**2.835**

EXAMPLE V (continued)

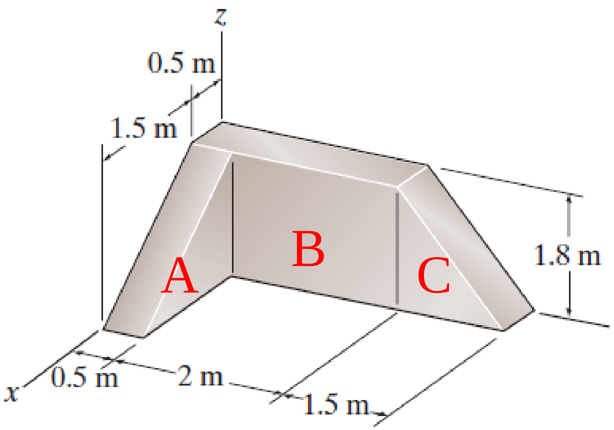


Table Summary

$V\ (m^3)$	$\tilde{x}\,V\ (m^4)$	$\tilde{y}\,V\ (m^4)$	$\tilde{z}\,V\ (m^4)$
3.6	**1.406**	**5.007**	**2.835**

Substituting into the Center of Volume equations:

$$\bar{x} = (\Sigma\,\tilde{x}\,V) / (\Sigma V) = 1.406 / 3.6 = 0.391\text{ m}$$

$$\bar{y} = (\Sigma\,\tilde{y}\,V) / (\Sigma V) = 5.007 / 3.6 = 1.39\text{ m}$$

$$\bar{z} = (\Sigma\,\tilde{z}\,V) / (\Sigma V) = 2.835 / 3.6 = 0.788\text{ m}$$