AREA MOMENT OF INERTIA

Objectives:

- Define the moments of inertia (MoI) for an area.
- Determine the MoI for an area by integration.
- Apply the parallel-axis theorem.
- Determine the moment of inertia (MoI) for a composite area.

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APPLICATIONS

Many structural members like beams and columns have cross sectional shapes like an I, H, C, etc..

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

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APPLICATIONS (continued)

Many structural members are made of tubes rather than solid squares or rounds.

Why?

This section of the book covers some parameters of the cross sectional area that influence the designer's selection.

Do you know how to determine the value of these parameters for a given cross-sectional area?

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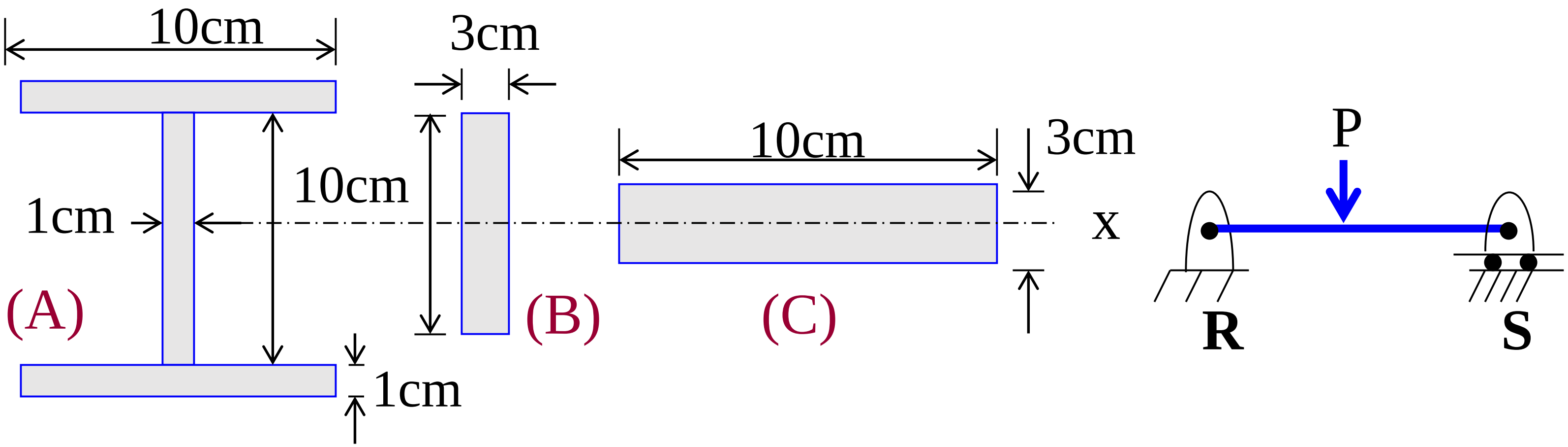
DEFINITION OF MOMENTS OF INERTIA FOR AREAS



Consider three different possible cross-sectional shapes and areas for the beam RS. All have the same total area and, assuming they are made of same material, they will have the same mass per unit length.

For the given vertical loading P on the beam shown on the right, which shape will develop less internal stress and deflection? Why?

It turns out that Section A has the highest MoI because most of the area is farthest from the x axis.

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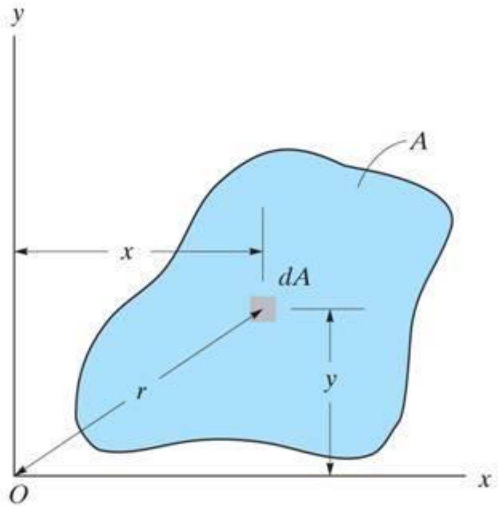
DEFINITION OF MOMENTS OF INERTIA FOR AREAS



For the differential area dA, shown in the figure:

$$d\,I_x = y^2\,dA\,,$$

$$d\,I_y = x^2\,dA\,, \text{ and,}$$

$$d\,J_O = r^2\,dA\,,$$ where J_O is the polar moment of inertia about the pole O or z axis.

The moments of inertia for the entire area are obtained by integration.

$$I_x = \int_A y^2\, dA\;;\quad I_y = \int_A x^2\, dA$$

$$J_O = \int_A r^2\, dA = \int_A (x^2 + y^2)\, dA = I_x + I_y$$

The MoI is also referred to as the second moment of an area and has units of length to the fourth power (m^4 or in^4).

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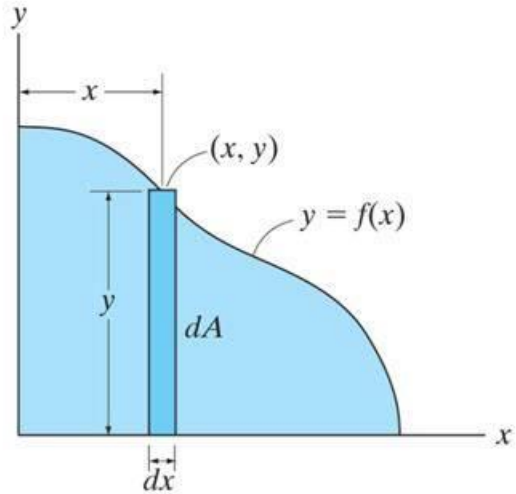
MoI FOR AN AREA BY INTEGRATION



For simplicity, the area element used has a differential size in only one direction (dx or dy). This results in a single integration and is usually simpler than doing a double integration with two differentials, i.e., dx·dy.

The step-by-step procedure is:

1. Choose the element dA: There are two choices: a vertical strip or a horizontal strip. Some considerations about this choice are:

   a) The element parallel to the axis about which the MoI is to be determined usually results in an easier solution. For example, we typically choose a horizontal strip for determining I_x and a vertical strip for determining I_y.

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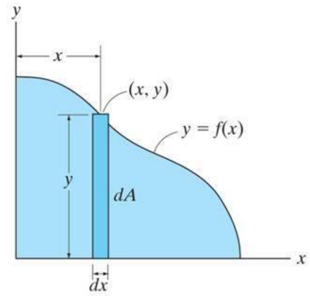
MoI FOR AN AREA BY INTEGRATION (continued)



b) If y is easily expressed in terms of x (e.g., $y = x^2 + 1$), then choosing a vertical strip with a differential element dx wide may be advantageous.

2. Integrate to find the MoI. For example, given the element shown in the figure above:

$I_y = \int x^2 \, dA = \int x^2 \, y \, dx$ and

Since the differential element is dx, y needs to be expressed in terms of x and the integral limit must also be in terms of x.

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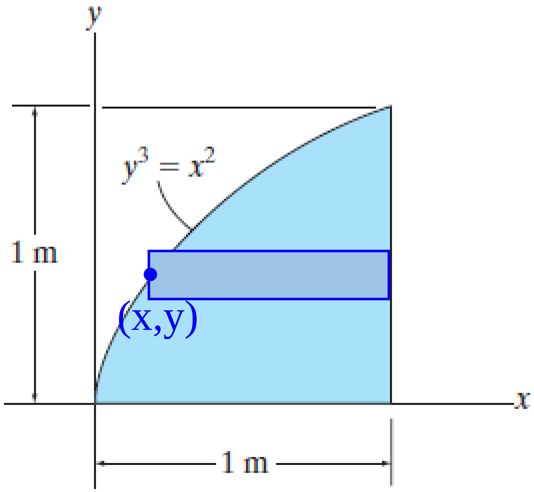
EXAMPLE I



Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.

Solution:

$$I_x = \int y^2 \, dA$$

$$dA = (1 - x)\, dy = (1 - y^{3/2})\, dy$$

$$I_x = \int_0^1 y^2 (1 - y^{3/2})\, dy$$

$$= \left[(1/3)\, y^3 - (2/9)\, y^{9/2} \right]_0^1 = 0.111 \text{ m}^4$$

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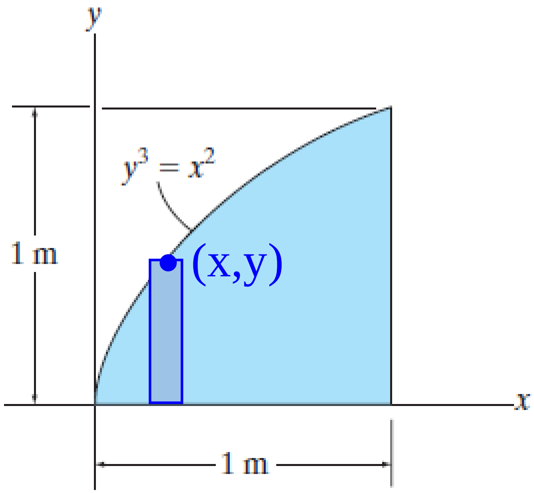
EXAMPLE I (continued)



$$I_y = \int x^2 \, dA = \int x^2 \, y \, dx$$

$$= \int x^2 \, (x^{2/3}) \, dx$$

$$= \int_0^1 x^{8/3} \, dx$$

$$= \left[(3/11) \, x^{11/3} \right]_0^1$$

$$= \underline{0.273 \text{ m}^4}$$

In the above example, I_x can be also determined using a vertical strip.

Then $I_x = \int (1/3) \, y^3 \, dx = \int_0 (1/3) \, x^2 \, dx = 1/9 = \underline{0.111 \text{ m}^4}$

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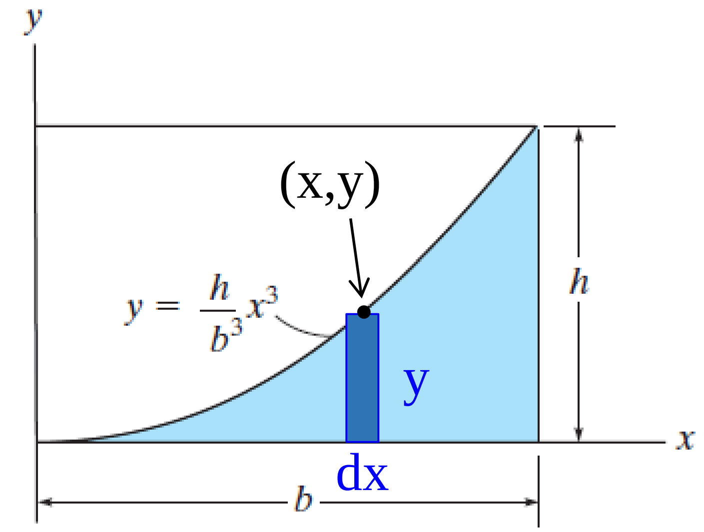
EXAMPLE II

Given: The shaded area shown.

Find: I_x and I_y of the area.



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EXAMPLE II (continued)

Solution:

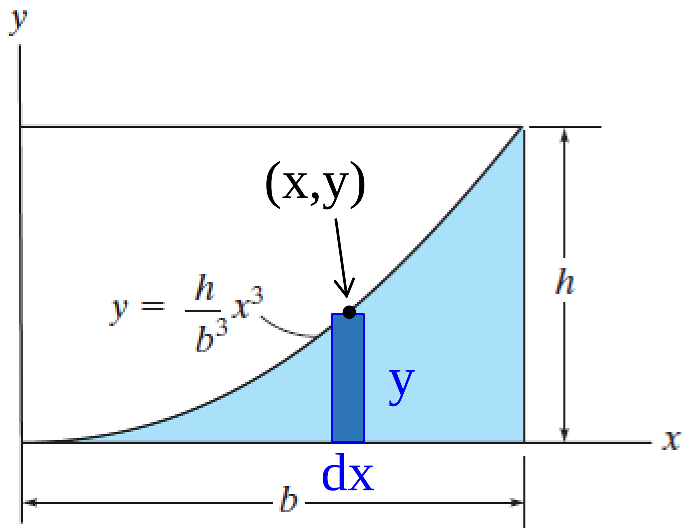
The moment of inertia of the rectangular differential element about the x-axis is $dI_x = (1/3)\, y^3\, dx$.

$$dI_x = \frac{1}{3}(y)^3 dx = \frac{1}{3}\left(\frac{h}{b^3}x^3\right)^3 dx$$

$$I_x = \int dI_x$$

$$= \int_{-2}^{0}\left(\frac{h^3}{3b^9}x^9\right) dx = \frac{h^3}{3b^9}\left(\frac{x^{10}}{10}\right)\Bigg|_0^b$$

$$I_x = \frac{1}{30} b h^3$$



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EXAMPLE II (continued)

The moment of inertia about the y-axis

$$I_y = \int x^2 dA$$

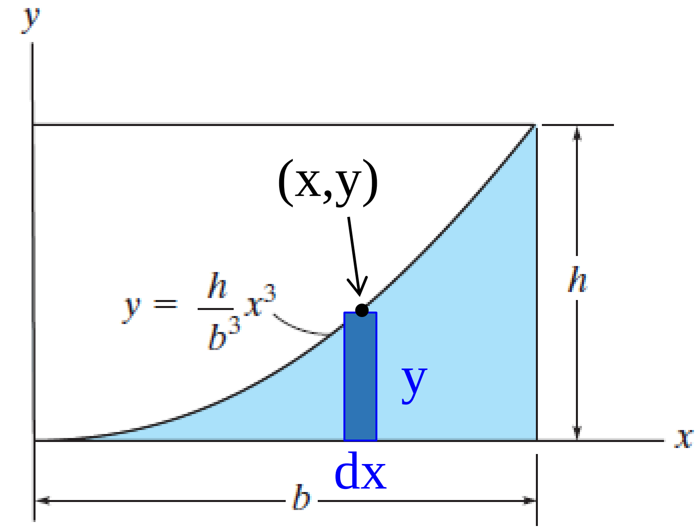
where

$$dA = y\,dx = \frac{h}{b^3}x^3 dx$$

$$I_y = \int_0^b x^2\left(\frac{h}{b^3}x^3\right)dx = \int_0^b \frac{h}{b^3}x^5 dx$$

$$= \frac{h}{b^3}\left(\frac{x^6}{6}\right)\Bigg|_0^b$$

$$I_y = \frac{1}{6}b^3 h$$



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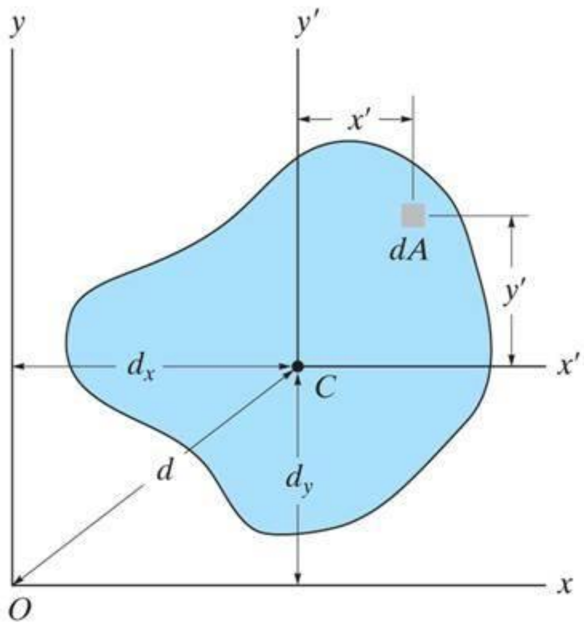
PARALLEL-AXIS THEOREM



This theorem relates the moment of inertia (MoI) of an area about an axis passing through the area's centroid to the MoI of the area about a corresponding parallel axis. This theorem has many practical applications, especially when working with composite areas.

Consider an area with centroid C. The x' and y' axes pass through C. The MoI about the x-axis, which is parallel to, and distance d_y from the x ' axis, is found by using the parallel-axis theorem.

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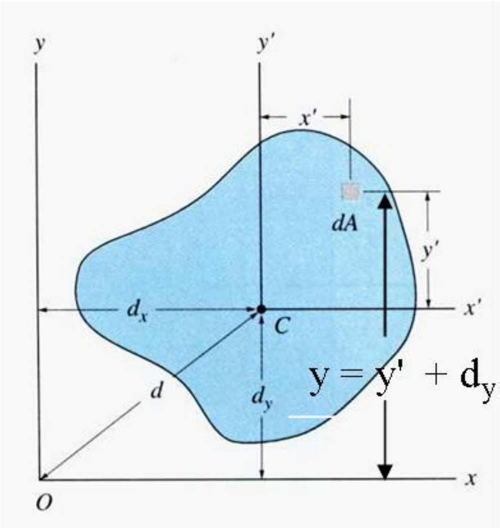
PARALLEL-AXIS THEOREM (continued)



$$I_x = \int_A y^2 \, dA = \int_A (y' + d_y)^2 \, dA$$

$$= \boxed{\int_A y'^{\,2} \, dA} + \boxed{2\, d_y \int_A y' \, dA} + \boxed{d_y^{\,2} \int_A dA}$$

Using the definition of the centroid, since C is the centroid, hence $\int_A y' \, dA = 0$.

Thus $I_X = \bar{I}_X' + A\, d_y^{\,2}$

Similarly, $I_Y = \bar{I}_Y' + A\, d_X^{\,2}$ and

$$J_O = \bar{J}_C + A\, d^2$$

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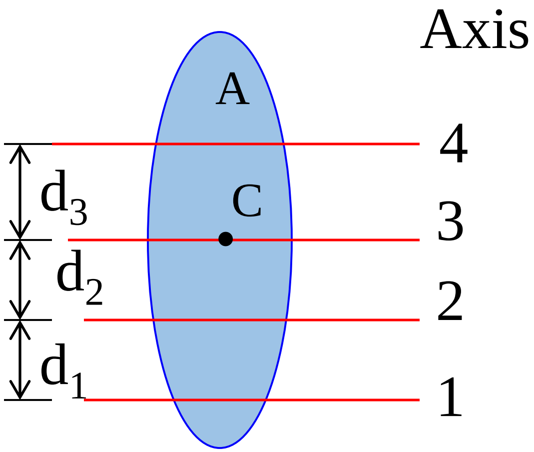
CONCEPT QUIZ

1. For the area A, we know the centroid's (C) location, area, distances between the four parallel axes, and the MoI about axis 1. We can determine the MoI about axis 2 by applying the parallel axis theorem ___ .



A) Directly between the axes 1 and 2.

B) Between axes 1 and 3 and then between the axes 3 and 2.

C) Between axes 1 and 4 and then axes 4 and 2.

D) None of the above.

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CONCEPT QUIZ (continued)

2. For the same case, consider the MoI about each of the four axes. About which axis will the MoI be the smallest number?

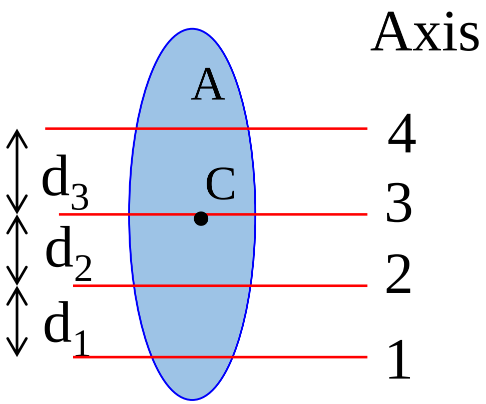
   A) Axis 1

   B) Axis 2

   C) Axis 3

   D) Axis 4

   E) Can not tell.



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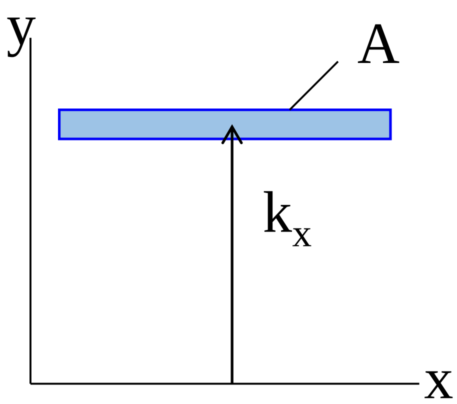
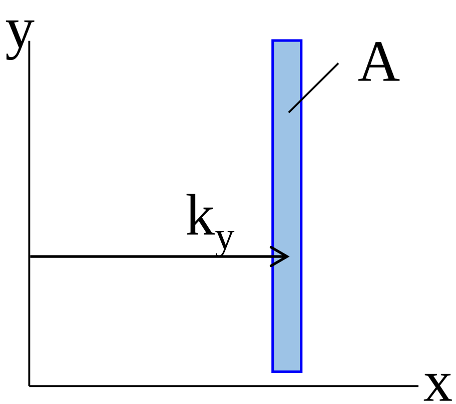
RADIUS OF GYRATION OF AN AREA





For a given area A and its MoI, I_x , imagine that the entire area is located at distance k_x from the x axis.

Then, $I_x = k_x^2 A$ or $k_x = \sqrt{(I_x / A)}$. This k_x is called the radius of gyration of the area about the x axis.

Similarly;

$$k_y = \sqrt{(I_y / A)} \quad \text{and} \quad k_O = \sqrt{(J_O / A)}$$

The radius of gyration has units of length and gives an indication of the spread of the area from the axes. This characteristic is important when designing columns.

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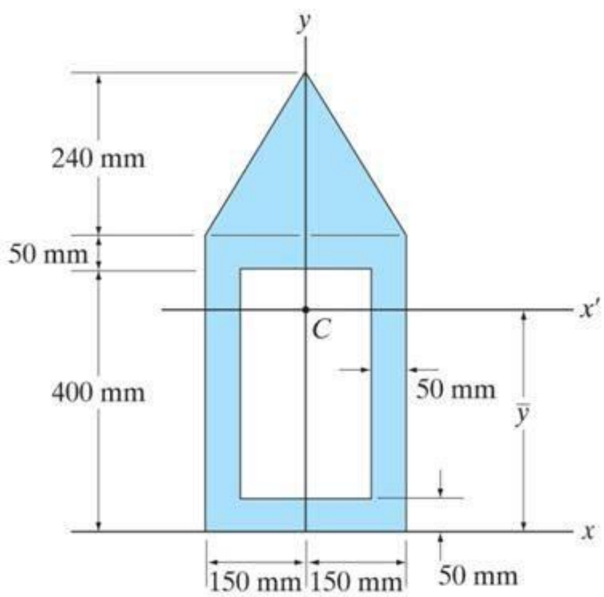
MOMENT OF INERTIA FOR A COMPOSITE AREA



A composite area is made by adding or subtracting a series of "simple" shaped areas like rectangles, triangles, and circles.

For example, the area on the left can be made from a rectangle plus a triangle, minus the interior rectangle.

The MoI about their centroidal axes of these "simpler" shaped areas are found in most engineering handbooks, with a sampling inside the back cover of the textbook.

Using these data and the parallel-axis theorem, the MoI for a composite area can easily be calculated.

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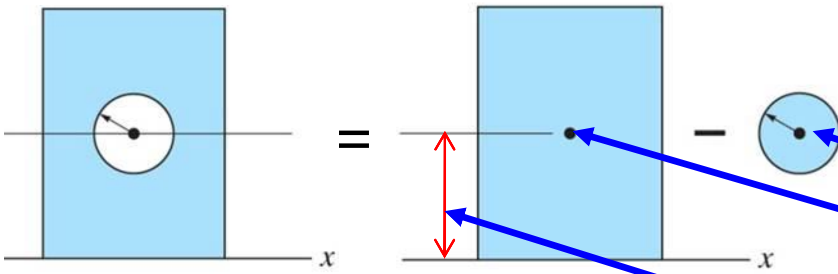
STEPS FOR ANALYSIS



1. Divide the given area into its simpler shaped parts.
2. Locate the centroid of each part and indicate the perpendicular distance from each centroid to the desired reference axis.
3. Determine the MoI of each "simpler" shaped part about the desired reference axis using the parallel-axis theorem ($I_X = \overline{I}_{X'} + A(d_y)^2$) .
4. The MoI of the entire area about the reference axis is determined by performing an algebraic summation of the individual MoIs obtained in Step 3. (Please note that the MoI of the hole is subtracted).

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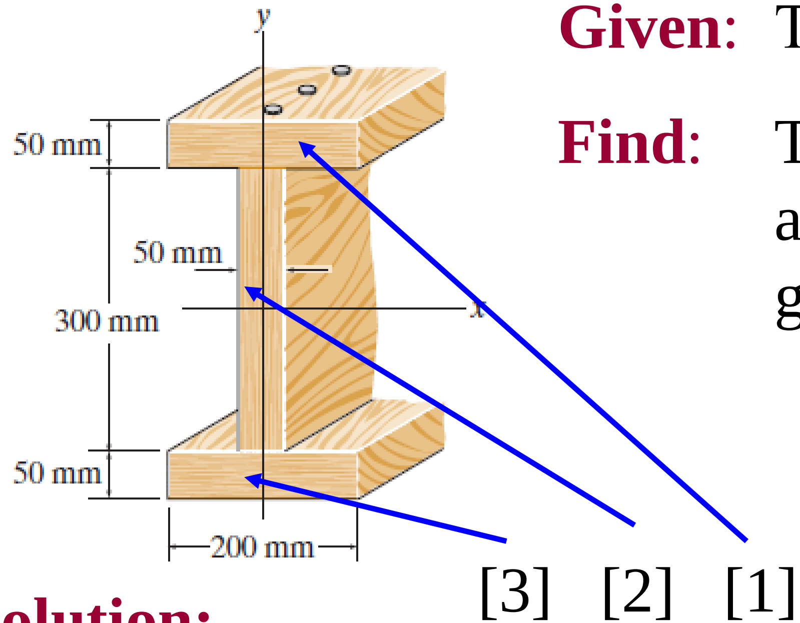
EXAMPLE III



Given: The beam's cross-sectional area.

Find: The moment of inertia of the area about the x-axis and the radius of gyration, k_x.

Solution:

1. The cross-sectional area can be divided into three rectangles ([1], [2], [3]) as shown.
2. The centroids of these three rectangles are in their center. The distances from these centers to the x-axis are 175 mm, 0 mm, and -175 mm, respectively.

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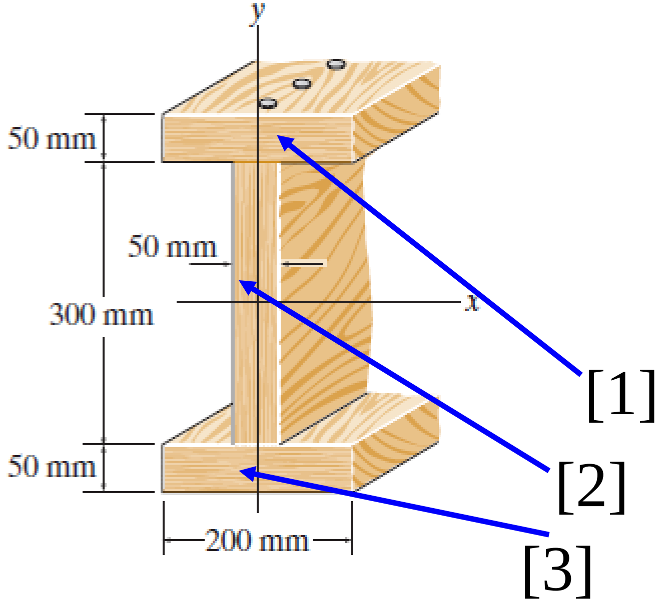
EXAMPLE III (continued)



3. From the inside back cover of the book, the MoI of a rectangle about its centroidal axis is (1/12) b h^3.

$$I_{x[2]} = (1/12)\,(50\text{ mm})\,(300\text{ mm})^3$$

$$= 1.125\times10^8\ \text{mm}^4$$

Using the parallel-axis theorem,

$$I_{x[1]} = I_{x[3]} = \bar{I}_{x'} + A\,(d_y)^2$$

$$= (1/12)\,(200)\,(50)^3 + (200)\,(50)\,(175)^2$$

$$= 3.083\times10^8\ \text{mm}^4$$

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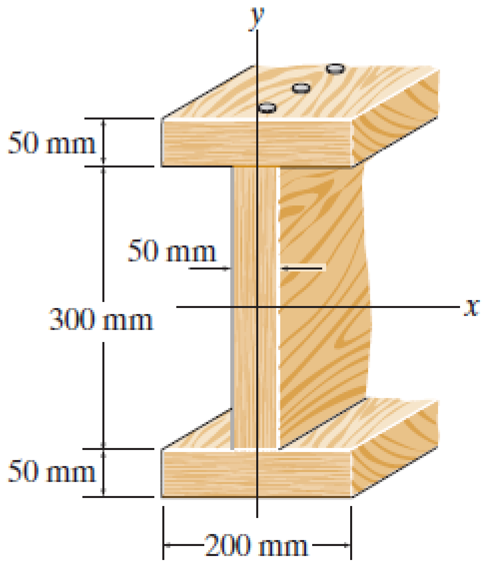
EXAMPLE III (continued)



Summing these three MoIs:

4. $I_x = I_{x1} + I_{x2} + I_{x3}$

$\underline{I_x = 7.291 \times 10^8 \ mm^4}$

Now, finding the radius of gyration:

$k_x = \sqrt{(I_x / A)}$

$A = 50\,(300) + 200\,(50) + 200\,(50) = 3.5 \times 10^4 \ mm^2$

$k_x = \sqrt{(7.291 \times 10^8) / (3.5 \times 10^4)} = \underline{144 \ mm}$

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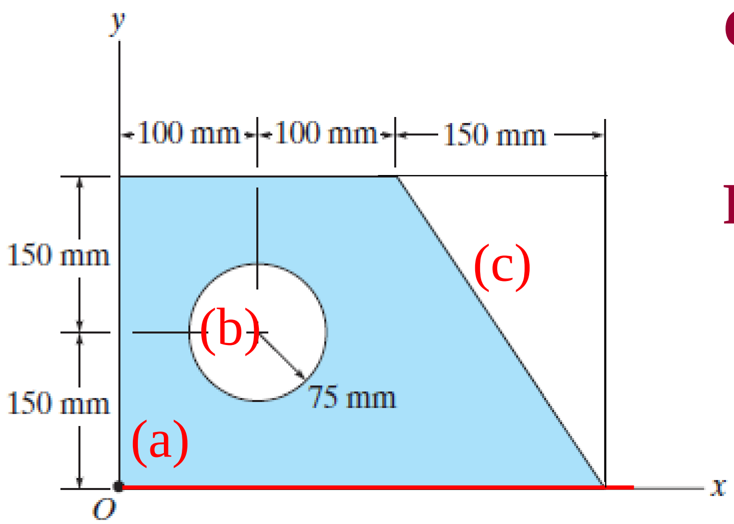
EXAMPLE IV



Given: The shaded area as shown in the figure.

Find: The moment of inertia for the area about the x-axis and the radius of gyration, k_X.

Solution:

1. The given area can be obtained by subtracting the circle (b) and the triangle (c) from the rectangle (a).

2. Information about the centroids of the simple shapes can be obtained from the inside back cover of the textbook.
   The perpendicular distances of the centroids from the x-axis are $d_a = 150$ mm, $d_b = 150$ mm, and $d_c = 200$ mm.

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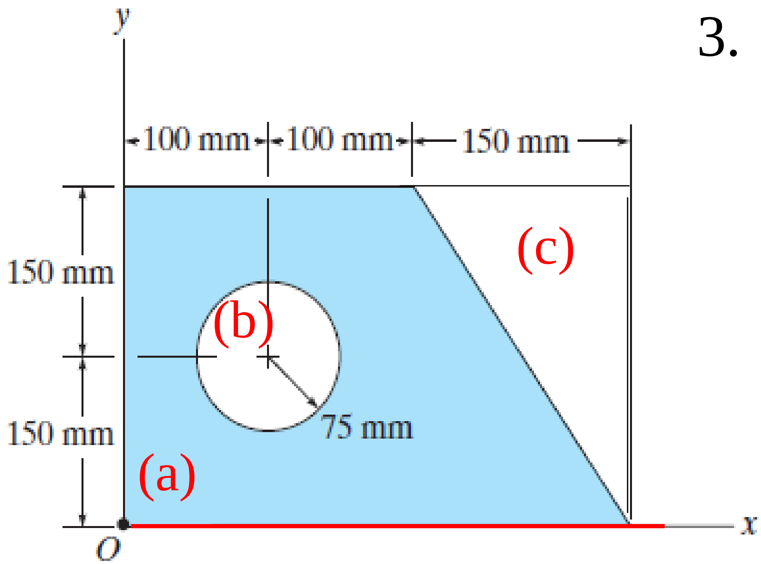
EXAMPLE IV (continued)



3. $I_{Xa} = (1/12)\,(350)\,(300)^3 + (350)(300)(150)^2$

$= 3.15\times10^9 \text{ mm}^4$

$I_{Xb} = (1/4)\,\pi\,(75)^4 + \pi(75)^2\,(150)^2$

$= 4.224\times10^8 \text{ mm}^4$

$I_{Xc} = (1/36)\,(150)\,(300)^3$

$+ (1/2)(150)\,(300)\,(200)^2$

$= 1.013\times10^9 \text{ mm}^4$

Summing: $I_X = I_{Xa} - I_{Xb} - I_{Xc}$

$= \underline{1.715\times10^9 \text{ mm}^4}$

The radius of gyration:

$k_X = \sqrt{(I_X / A)}$

$A = 350\,(300) - \pi\,(75)^2 - (1/2)\,150\,(300) = 8.071\times10^4 \text{ mm}^2$

$k_X = \sqrt{1.715\times10^9 / 8.071\times10^4} = 146 \text{ mm}$

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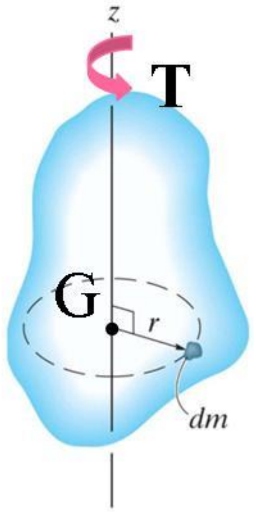
MASS MOMENT OF INERTIA



Consider a rigid body with a center of mass at G. It is free to rotate about the z axis, which passes through G. Now, if we apply a torque T about the z axis to the body, the body begins to rotate with an angular acceleration α.

T and α are related by the equation $T = I\alpha$. In this equation, I is the mass moment of inertia (MMI) about the z axis.

The MMI of a body is a property that measures the resistance of the body to angular acceleration. This is similar to the role of mass in the equation F = m a. The MMI is often used when analyzing rotational motion (in dynamics course).

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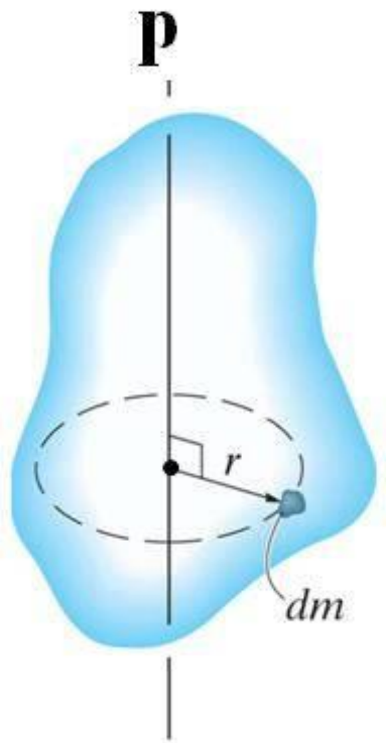
DEFINITION OF THE MMI



Consider a rigid body and the arbitrary axis p shown in the figure. The MMI about the p axis is defined as $I = \int_m r^2\,dm$, where r, the "moment arm," is the perpendicular distance from the axis to the arbitrary element dm.

The MMI is always a positive quantity and has a unit of $kg \cdot m^2$ or $slug \cdot ft^2$.

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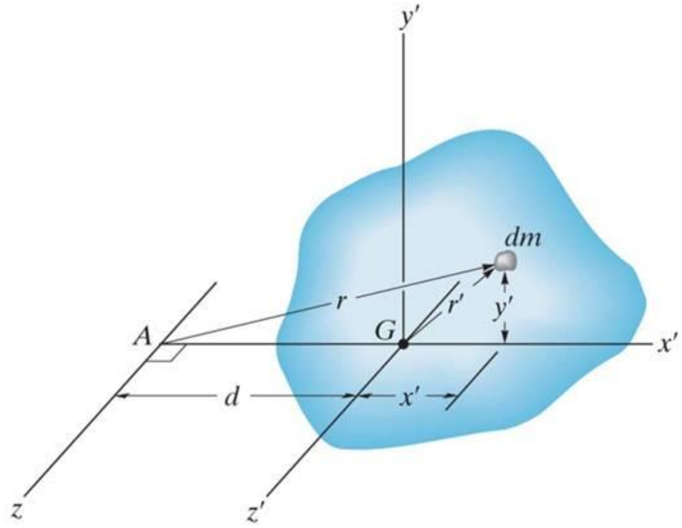
RELATED CONCEPTS

Parallel-Axis Theorem

Just as with the MoI for an area, the parallel-axis theorem can be used to find the MMI about a parallel axis z that is a distance d from the z' axis through the body's center of mass G. The formula is $I_z = I_G + (m)(d)^2$ (where m is the mass of the body).



The radius of gyration is similarly defined as
$k = \sqrt{(I / m)}$

Finally, the MMI can be obtained by integration or by the method for composite bodies. The latter method is easier for many practical shapes.

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